Sr. No. 003101

December 2024

B. Tech. 1st SEMESTER,

Mathematics-I (Calculus and Linear Algebra) (BSC-103E / BSCH-103E)

Time: 3 Hours Max. Marks: 75

Instructions:
1. *It is compulsory to answer all the questions (1.5 marks each) of Part -A in short.*
2. *Answer any four questions from Part -B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART -A

Q1 (a) Evaluate (1.5)

$$\int_0^1 x^{11}(1-x)^5\,dx.$$

(b) Prove that $\Gamma(n+1) = n\,\Gamma(n)$, where $n > 0$. (1.5)

(c) State Rolle's theorem. Give its geometric interpretation. (1.5)

(d) Evaluate (1.5)

$$\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}.$$

(e) Find the rank of the matrix (1.5)

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}.$$

(f) Find (1.5)

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

(g) Let $V = \mathbb{R}^2 = \{(a,b): a, b \in \mathbb{R}\}$. Show that $W = \{(a,0): a \in \mathbb{R}\}$ is a subspace of V. (1.5)

(h) Test whether the map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x,0,0)$ for all $(x,y,z) \in \mathbb{R}^3$ is linear. (1.5)

(i) Find the eigenvalues of the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$. (1.5)

(j) If $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$, then find the eigenvalues of $A^2 - 2A + I$. (1.5)

PART -B

Q2 (a) Find the equation of the evolute of the parabola $y^2 = 4ax$. (7)

(b) Find the volume generated when the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$ is revolved about the y-axis. (8)

Q3 (a) Evaluate (7)

$$\lim_{x \to 0}\left(\frac{a^x + b^x + c^x + d^x}{4}\right)^{1/x}$$

where a, b, c and d are positive numbers.

(b) Show that (8)

$$\log_e(1 + e^x) = \log_e 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots\cdots$$

and hence deduce that

$$\frac{e^x}{1 + e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots\cdots$$

Q4 (a) Solve the following system of equations (7)

$$\begin{aligned} x + y + z &= 4 \\ x - y + z &= 0 \\ 2x + y + z &= 5 \end{aligned}$$

(b) For what values of k, the equations (8)

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= k \\ 4x + y + 10z &= k^2 \end{aligned}$$

have

(i) a unique solution,
(ii) infinitely many solutions,
(iii) no solution.

Q5 (a) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined on the basis $T(1, 0, 0) = (0, 0)$, $T(0, 1, 0) = (1, 1)$ and $T(0, 0, 1) = (1, -1)$. Compute $T(4, -1, 1)$. Determine the nullity and rank of T. (7)

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(b) Find the matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by (8)

$$T(x, y) = (2x - y, 3x + 4y, x)$$

with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Q6 (a) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$. (7)

(b) Find an orthonormal basis of the inner product space $\mathbb{R}^3$ over $\mathbb{R}$ with standard inner product, given the basis $B = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$ using Gram-Schmidt orthogonalization process. Also, find the Fourier coefficients of the vector $(1, 1, 2)$ relative to orthonormal basis. (8)

$b_3 = \cancel{(3, -5, -3)}$ $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$
$b_2 = \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$

Q7 (a) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. (7)

(b) Diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by means of an orthogonal transformation. (8)

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