Dec. 2025

B.Tech. (Mathematics-I) (First Semester)

Calculus and Linear Algebra

(MTU-149-V/BSC-103E/BSCH-103E)

Time : 3 *Hours*] [*Maximum Marks* : 75

Note : It is compulsory to answer all the questions (1.5 marks each) of Part A in short. Answer any *four* questions from Part B in detail. Different sub-parts of a question are to be attempted adjacent to each other.

Part A

1. (a) Evaluate : 1.5

$$\int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta.$$

(b) Evaluate : **1.5**

$$\int_1^{\infty} \frac{dx}{x^2}, \text{ if it exists.}$$

(c) Verify Cauchy's Mean Value Theorem for $f(x) = x^3$, $g(x) = x^2$ in $[1, 2]$. **1.5**

(d) Evaluate : **1.5**

$$\lim_{x \to \infty} \frac{x^n}{e^x}.$$

(e) Evaluate : **1.5**

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

(f) Define the rank of a matrix. Give an example. **1.5**

(g) Let $V = \mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$. Examine whether the subset $W = \{(a, b) : 2a + 3b = 2; a, b \in \mathbb{R}\}$ is a subspace of V or not. **1.5**

(h) Examine whether the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, 0, 0)$ for all $(x, y, z) \in \mathbb{R}^3$ is linear or not. 1.5

(i) Show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix. 1.5

(j) Let V be an inner product space over a field F. Prove that for all $u, v, w \in V$ and $\alpha, \beta \in F$ $< u, \alpha v + \beta w > = \overline{\alpha} < u, v > + \overline{\beta} < u, w >$. 1.5

Part B

2. (a) Find the equation of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ 7

(b) The area bounded by $y^2 = 4x$ and the line $x = 4$ is revolved about the line $x = 4$. Find the volume of the solid of revolution. 8

3. (a) If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite, then find the value of a and the limit. 7

(b) If $0 < a < b$, then prove that : 8

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Hence, deduce that :

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

4. (a) Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to normal form and hence find the rank. 7

(b) Test for consistency and solve : 8

$$x + 2y + z = 2$$

$$2x - y - z = 2$$

$$4x - 7y - 5z = 2$$

5. (a) Find the dimension of the subspace spanned by (1, 1, 2, 4), (2, −1, − 5, 2), (1,− 1,− 4, 0) and (2, 1, 1, 6) in $\mathbb{R}^4$. 7

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. 8

(i) Determine T.

(ii) Find $T(2, 5)$.

(iii) Find the rank of T and nullity of T.

(iv) Determine whether T is one-one or onto.

6. (a) Find the eigen values and eigen vectors of the matrix : 7

$$A = \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}.$$

(b) Let $V = P(\mathbb{R})$ be the vector space of polynomials over $\mathbb{R}$ with inner product defined by $\langle f, g \rangle = \int_0^1 f(t) g(t) dt$. Let $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$. Find $\langle f, g \rangle$, $\|f\|$ and $\|g\|$. 8

7. (a) Using Taylor's theorem, prove that : 7

$$x-\frac{x^3}{6}<\sin x<x-\frac{x^3}{6}+\frac{x^5}{120} \quad \text{for } x>0.$$

(b) Find an orthonormal basis of the inner product space $\mathbb{R}^3$ over $\mathbb{R}$ with the standard inner product, given the basis $\mathcal{B}=\{(1, 0, 1), (1, 0, -1), (0, 3, 4)\}$ using Gram Schmidt orthogonalization process. 8

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