2 Combinatorics

LEARNING OBJECTIVES

After reading this chapter you will be able to understand the following:

- The permutations and combinations by which sets are organized to use the data they contain and to interpret them
- Basic concepts of probability consisting of discrete, conditional, random variables, and their components
- About recursion and its use to define sets and sequences
- The use of recursive style for solving difference equation (recurrence relations) and the discretization of differential equations
- About the inclusion–exclusion principle and its use to count the number of elements in a union of sets, and to solve counting problems
- About generating functions and their use to solve recurrence relations

2.1 INTRODUCTION

Combinatorics deals with the study of arrangements of objects. It is an important part of discrete mathematics. Enumeration, the counting of objects to solve a variety of problems, is also a key part of combinatorics. Counting is used to determine the complexity of algorithms, to determine sufficient telephone numbers, or Internet protocol addresses to meet demand. Moreover, counting techniques are useful in computing probabilities of events.

The basic rules of counting, which we will study in this chapter, can solve different types of problems. Another important combinatorial tool is the pigeonhole principle, which we will discuss in this chapter. Here, we will also study the recurrence relations, a tool for the analysis of computer programs. In addition, the generating function, inclusion–exclusion principle, and its applications will also be discussed here.

2.2 BASIC PRINCIPLES OF COUNTING

Counting problems exist throughout mathematics and computer science. Sometimes it is necessary to count the successful outcomes of experiments and all their possible outcomes to determine probabilities of discrete events. It is also needed to count the number of operations used by an algorithm to study its time complexity. Here, we will introduce the basic techniques of counting, the methods of which serve as the foundation for almost all counting techniques.

The basic principles of counting, such as multiplication principle and addition principle are described below.

2.2.1 Multiplication Principle (the Principle of Sequential Counting)

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then the total number of occurrence of the events E and F in the given order is mn. More generally, suppose an event E_1 can occur in n_1 ways, and, following E_1, a second event E_2 can occur in n_2 ways, and following E_2, a third event E_3 can occur in n_3 ways, and so on. Then the total number of occurrence of the events $E_1, E_2, E_3, \ldots,$ in the order indicated is $n_1 n_2 n_3 \cdots$.

The multiplication principle is also named as the *fundamental principle of counting*.

Note For AND → '×' (multiply)

Example 2.1 Find the number of four-letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in four vacant places by the four letters, keeping in mind that the repetition is not allowed. The first place can be filled in four different ways by any of the four letters R, O, S, and E. Following which, the second place can be filled in by any of the remaining three letters in three different ways, the third place in two different ways, and the fourth place in one way. Thus, the number of ways in which the four places can be filled, by the multiplication principle (or, principle of counting) is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Note If the repetition of the letters was allowed, how many words can be formed? Each of the four vacant places can be filled in succession in four different ways. Hence, the required number of words is $4 \times 4 \times 4 \times 4 = 256$.

Example 2.2 Given four flags of different colours, how many different signals can be generated if a signal requires the use of two flags one below the other?

Solution There will be as many signals as there are ways of filling in two vacant places in succession by the four flags of different colours. The upper vacant place can be filled in four different ways by any of the four flags; following which, the lower vacant place can be filled in three different ways by any of the remaining three different flags. Hence, by multiplication principle, the required number of signals is $4 \times 3 = 12$.

2.2.2 Addition Rule (the Principle of Disjunctive Counting)

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both the events cannot occur simultaneously. Then, E or F can occur in $m + n$ ways. More generally, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, a third event E_3 can occur in n_3 ways, $\ldots$, and suppose no two of the events can occur at the same time. Then, one of the events can occur in $n_1 + n_2 + n_3 + \cdots$ ways.

Note For OR → '+' (addition)

Example 2.3 Suppose there are eight male professors and five female professors teaching a history class. In how many ways a student can choose a history professor?

Solution A student can choose a history professor in $8 + 5 = 13$ ways.

Example 2.4 In how many ways can we get a sum of 7 or 11 when two distinguishable dice are rolled?

Solution The ordered pairs in which the sum is 7 are (1,6), (2,5), (3,4), (4,5), (5,2), (6,1), which are distinct. There are six ways to obtain the sum 7. Similarly, the ordered pairs in which the sum is 11 are (5,6), (6,5), which are also distinct. The number of ways in which we get a sum 11 with the two dice is 2. Thus, we can get a sum 7 or 11 with two distinguishable dice in $6 + 2 = 8$ ways.

2.3 FACTORIAL NOTATION

If n is a natural number, then the product of all the natural numbers from 1 to n is called 'n-factorial'. It is denoted by the symbol $n!$ or $\lfloor n$.

From the definition,

$$n! = n(n-1)(n-2)\cdots 3.2.1$$

The factorial notation $n!$ can also be defined recursively as follow:

$$0! = 1, (n+1)! = n!(n+1), n \geq 0$$

From the above recursive definition, we get

$$1! = 0!(1) = 1, \quad 2! = 1!(2) = 1.2, \quad 3! = 2!(3) = 1.2.3$$

Example 2.5 Find the value of 8!.

Solution For $n \geq 0$, $(n+1)! = n!(n+1)$

Hence,

$$\begin{aligned} 8! &= (7+1)! = 7!\,8 = 6!\,7.8 = 5!\,6.7.8 \\ &= 4!\,5.6.7.8 = 3!\,4.5.6.7.8 \\ &= 2!\,3.4.5.6.7.8 = 1!\,2.3.4.5.6.7.8 \\ &= 1.2.3.4.5.6.7.8 \\ &= 40{,}320 \end{aligned}$$

Example 2.6 Simplify

(i) $\dfrac{n!}{(n-1)!}$

(ii) $\dfrac{(n+1)!}{n!}$

.5 PERMUTATIONS (ARRANGEMENTS OF OBJECTS)

Any arrangement of a set of n objects in a definite order is called a *permutation* of the objects, taken all at a time. Any arrangement of any r of these n objects ($r \leq n$) in a definite order is called an *r-permutation* or a permutation of the n objects taken r at a time.

For example Consider the set of letters *a*, *b*, *c*, and *d*. Then

(i) *bdca*, *dcba*, and *acdb* are permutations of the four letters, taken all at a time
(ii) *bad*, *adb*, *cbd*, and *bca* are permutations of the four letters taken three at a time
(iii) *ad*, *cb*, *da*, and *bd* are permutations of the four letters taken two at a time

The number of permutations of n different objects taken r at a time is denoted by ^nP_r.

Theorem 2.3 The number of r = permutations of a set with n distinct elements not responding

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof There will be as many permutations as there are ways of filling in r vacant places by n different objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways, ..., the rth place can be filled in $\{n(n-1)\}$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\cdots\{n-(r-1)\}$ or $n(n-1)(n-2)\cdots(n-r+1)$.

Thus, by the fundamental principle of counting (multiplication rule), there are $n(n-1)(n-2)\cdots(n-r+1)$, r − permutations of the set, i.e.,

$$\begin{aligned} ^nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{[n(n-1)(n-2)\cdots(n-r+1)][(n-r)(n-r-1)\cdots 2.1]}{[(n-r)(n-r-1)\cdots 2.1]} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Hence,

$$^nP_r = \frac{n!}{(n-r)!}$$

Note The number of permutations of n distinct objects taken n at a time is

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Again, consider three letters a, b, c. Then, there are $3! = 3.2.1 = 6$ permutations of the three letters, such as, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

EXAMPLE 2.10 How many words of three distinct letters can be formed from the letters of the word LAND?

Solution The number of three distinct letters can be formed from the four letters of the word LAND is

$$^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$$

2.5.1 Permutations with Repetitions

Theorem 2.4 The number of permutations of n objects, of which n_1, are alike of one kind, n_2 are alike of another kind, n_3 are alike of third kind, ..., n_r are alike of rth kind, is

$$^nP_{n_1, n_2, n_3, \ldots, n_r} = \frac{n!}{n_1! n_2! n_3! \cdots n_r!}$$

where $n = n_1 + n_2 + n_3 + \cdots + n_r$.

Proof Let the required number of permutations be x. If the n_1 like objects are unlike, then for each of these x arrangements, the n_1 like objects can be rearranged among themselves in $n_1!$ ways, without altering the positions of the other objects.

So, the number of permutations will be $xn_1!$. Similarly, if n_2 like objects are unlike, each of these $xn_1!$ permutations will give rise to $n_2!$ permutations. Therefore, the number of permutations will be $xn_1!\ n_2!$ If all the objects are unlike, the number of permutations will be $n_1!\ n_2! \cdots n_r!$. But, if all the objects are unlike, the number of permutations with n objects will be $n!$. Hence,

$$xn_1!n_2! \cdots n_r! = n!$$

$$x = \frac{n!}{n_1!n_2!n_3! \cdots n_r!}$$

i.e.,

$$^nP_{n_1, n_2, n_3, \ldots, n_r} = \frac{n!}{n_1!n_2!n_3! \cdots n_r!}$$

Note The number of permutations of n distinct objects, taken r at a time (when repetitions are allowed) is $(n)^r$.

Example 2.11 If there are four black, three green, and five red balls, then in how many ways these colour can be arranged in a row?

Solution Here, the total number of balls $= 4 + 3 + 5 = 12$. Also, the black balls, green balls, and the red balls are all alike. Hence, the balls can be arranged in a row in

$$\frac{12!}{4!\ 3!\ 5!} = 27{,}720 \text{ ways}$$

Example 2.12 How many words can be formed using the letter A thrice, the letter B twice, and the letter C once?

Solution Given six objects $A, A, A, B, B,$ and C of which three are alike of the same kind, two are alike of another kind, and one is of its own kind. Hence, the total number of permutations of the requisite number of words formed

$$\frac{6!}{3!\ 2!\ 1!} = 60$$

EXAMPLE 2.13 Find how many arrangements can be made with the letters of the word 'MATHEMATICS'.

Solution There are 11 letters in the word 'MATHEMATICS'. Out of these letters M is repeated twice, A is repeated twice, T is repeated twice, and the rest are all different. So, the required number of arrangements

$$\frac{11!}{2!\,2!\,2!} = 4989600$$

EXAMPLE 2.14 How many four-digit numbers can be formed using the digits 2, 4, 6, and 8 when repetition of digits is allowed.

Solution Here, we have four digits. So,

Number of ways of filling unit's place = 4
Number of ways of filling ten's place = 4
Number of ways of filling hundred's place = 4
Number of ways of filling thousand's place = 4

Thus, the total number of four-digit numbers = 4^4 = 256.

2.5.2 Circular Permutations

Instead of arranging the objects in a line, if we arrange them in the form of a circle, we call them *circular permutations*. In circular permutations, what really matters is the position of an object relative to the others.

Suppose n persons $(a_1, a_2, \ldots, a_n)$ are to be arranged around a ring. There are $n!$ ways in which they can be arranged in a row; on the other hand, all the linear arrangements

$$\begin{array}{ll} a_1, a_2, a_3, a_4, \ldots, a_n; & a_2, a_3, a_4, \ldots, a_n, a_1 \\ a_n, a_1, a_2, \ldots, a_{n-1}; & a_{n-1}, a_n, a_1, a_2, \ldots, a_{n-2} \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \end{array}$$

will lead to the same arrangement in a ring. So, each circular arrangement corresponds to n linear arrangements. Hence, the total number of circular arrangements of n persons is

$$\frac{n!}{n} = (n-1)!$$

Thus, there are $(n - 1)!$ permutations of n distinct objects in a circle.

If we consider the *clockwise* and *anticlockwise* arrangements in the circular permutations, then the following propositions can be possible:

(i) When distinction is made between the clockwise and the anticlockwise arrangements of n different objects around a circle, then the number of arrangements = $(n - 1)$.

(ii) If no distinction is made between the clockwise and the anticlockwise arrangements of n different objects around a circle, then the number of arrangements = $(1/2)(n - 1)!$

EXAMPLE 2.15 In how many different ways can five men and five women sit around a table, if

(a) there is no restriction?

(b) no two women sit together?

Solution

(a) If there is no restriction then this problem is same as the circular permutation of 10 objects (five men and five women). So, the number of permutations are $(10-1)! = 9! = 362880$.

(b) If two women are not allowed to sit side by side, that is each woman should occupy a sit between two men.

The number of ways five men can sit around a round table $= (5-1)! = 4! = 24$. Once these five men have already occupied alternate seats, the five women can sit in the five empty seats in $5! = 5 \times 4 \times 3 \times 2 = 120$ ways. Thus, the total number of ways $= 24 \times 120 = 2880$.

Example 2.16 In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements?

Solution Seven persons can sit around a table in 6! ways. But in clockwise and anticlockwise arrangements, each person will have the same neighbour. So, the required number of ways $= (1/2) \times (6!) = 360$.

Example 2.17 Find the number of ways in which eight different beads can be arranged to form a necklace.

Solution Fixing the position of one bead, the remaining beads can be arranged in 7! ways. But there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements $= (1/2) \times (7!) = 2520$.

2.6 COMBINATIONS (SELECTION OF OBJECTS)

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a *combination*.

Suppose we want to select two out of three persons A, B, and C. We may choose AB or BC or AC. Clearly, AB and BA represent the same selection or group but they give rise to different arrangements. Clearly, in a groups or selection, the order in which the objects are arranged is immaterial.

For example

(i) The different combinations formed of three letters a, b, and c, taken two at a time, are ab, bc, and ca.

(ii) The only combination that can be formed of three letters a, b, and c taken all at a time, is abc.

(iii) Various groups of two out of four persons A, B, C, and D are AB, AC, AD, BC, BD, and CD.

Difference between a *permutation* and a *combination*: In a combination, only a group is made and the order in which the objects are arranged is immaterial. On the other hand, in a permutation, not only a group is formed, but also an arrangement in a definite order is considered

For example

(i) ab and ba are two different permutations, but each represents the same combination.

(ii) abc, acb, bac, bca, cab, and cba are six different permutations, but each one of them rep resents the same combination, namely a group of three objects a, b, and c.

Note We use the word 'arrangements' for permutations and 'selections' for combinations.

The number of combinations of n objects, taken r at a time, is denoted by ^nC_r. The symbol ^nC_r is defined only when n and r are integers such that $n \geq r$, $n > 0$, and $r \geq 0$.

2.6.1 Combinations of n Different Objects

Theorem 2.5 The number of combinations of n distinct objects, taken r at a time, is given by

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Proof Let the number of combinations of n objects, taken r at a time, be x. Then, $^nC_r = x$.

Now, each combination contains r objects, which may be arranged amongst themselves in $r!$ ways. Thus, each combination gives rise to $r!$ permutations. So, x combinations will give rise to $x(r!)$ permutations. Therefore, the number of permutations of n things, taken r at a time, is $x(r!)$.

Consequently, $^nP_r = x(r!) = {}^nC_r(r!)$. Thus,

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} \qquad \left(\because {}^nP_r = \frac{n!}{(n-r)!}\right)$$

Note We may write

$$^nC_r = \frac{n(n-1)(n-2)\ldots \text{ to } r \text{ factors}}{r!}$$

In the present topic, we concentrate on counting unordered selection of objects.

For example Consider a query posing like, how many different committees of three students can be formed with four students?

To answer this question, we need only to find the number of subsets with three elements that form the set containing four students. We see that there are four such subjects, one for each of the four students, because choosing four students is the same as choosing one of the four students to leave out of the groups. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen, is immaterial.

This example illustrates that many counting problems can be solved by finding the number of subsets of a particular size of a set with n elements, where n is a positive integer.

An r combination of elements of a set is an unordered selection of r elements from the set. Thus, an r combination is simply a subset of the set with r elements.

For example It is seen that $^4C_2 = 6$, because the two combinations of $\{a, b, c, d\}$ yield six subsets, such as $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

EXAMPLE 2.18 In how many ways a committee of five members can be selected from six men and five ladies, consisting of three men and two ladies?

Solution Three men out of six and two ladies out of five can be selected in

$$^6C_3 \times {}^5C_2 = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1}\right) \times \left(\frac{5 \times 4}{2 \times 1}\right) = 200 \text{ ways}$$

Example 2.19 How many committees of five people can be chosen from 20 men and 12 women

(a) if exactly three men must be on each committee?

(b) if at least three women must be on each committee?

Solution

(a) We choose three men from twenty men and then two women from twelve women. So, the number of committees will be

$$^{20}C_3 \times {}^{12}C_2 = 1140 \times 66 = 75240$$

(b) We choose at least three women here, that is, three women, four women, and five women are to be chosen in each case from twelve women. Then, by addition rule, the number of committees will be

$$^{12}C_3 \times {}^{20}C_2 + {}^{12}C_4 \times {}^{20}C_1 + {}^{12}C_5 \times {}^{20}C_0$$
$$= 220 \times 190 + 495 \times 20 + 792 \times 1 = 52492$$

Example 2.20 A collection of 10 electric bulbs contains 3 defective ones.

(a) In how many ways can a sample of four bulbs be selected?

(b) In how many ways can a sample of four bulbs be selected which contain two good bulbs and two defective ones?

(c) In how many ways can the sample of four bulbs be selected so that either the sample contains three good ones and one defective ones or one good and three defective ones?

Solution

(a) The four bulbs can be selected out of ten bulbs in

$$^{10}C_4 = \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210 \text{ ways}$$

(b) Two bulbs can be selected out of seven good bulbs in $^7C_2$ ways and two defective bulbs can be selected out of three defective bulbs in $^3C_2$ ways. Thus, the number of ways in which a sample of four bulbs containing two good bulbs and two defective bulbs can be selected as

$$^7C_2 \times {}^3C_2 = \frac{7!}{2!\,5!} \times \frac{3!}{2!\,1!} = \frac{7 \times 6}{2} \times 3 = 63$$

(c) Three good bulbs can be selected from seven good bulbs in $^7C_3$ ways and one defective bulb can be selected out of three defective ones in $^3C_1$ way.

Similarly, one good bulb can be selected from seven good bulbs in $^7C_1$ ways and three defective ones in $^3C_3$ ways.

So, the number of ways of selecting a sample of four bulbs containing three good ones and one defective or one good and three defective ones are

$$^7C_3 \times {}^3C_1 + {}^7C_1 \times {}^3C_3 = \frac{7!}{3!\,4!} \times \frac{3!}{1!\,2!} + \frac{7!}{1!\,6!} \times \frac{3!}{3!\,0!}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2} \times 3 + 7 = 35 \times 3 + 7 = 112$$

2.6.2 Combinations with Repetitions

When repetition of elements is allowed, the r combinations (with objects of size r) form a set with n elements which can be expressed as

$$^{n+r-1}C_r = (n + r - 1)!/(r!)(n - 1)! = {}^{n+r-1}C_{n-1}$$

We will illustrate the above combinations by the following examples.

Example 2.21 Consider a cookie shop in which there are four different kinds of cookies. Find the number of different ways of choosing six cookies (assuming only the type of cookies, and not the individual cookies or the order in which they are chosen).

Solution The number of ways to choose six cookies is the number of six combinations of a set with four elements, which is given by

$$^{4+6-1}C_6 = {}^9C_6 = {}^9C_3 = (9.8.7)/(1.2.3) = 84$$

Thus, there exists 84 different ways to choose the six cookies.

Example 2.22 Four boys picked up 30 mangoes. In how many ways can they divide them if all the mangoes be identical?

Solution Clearly, 30 mangoes can be distributed among four boys such that each boy can receive any number of mangoes. Hence, total number of ways $= {}^{30+4-1}C_{4-1} = {}^{33}C_3 = 5456.$

Example 2.23 Assume that a valid computer password consists of seven characters, the first of which is a letter selected from the set $\{A, B, C, D, E, F, G\}$ and the remaining characters are letters chosen from the English alphabet or a digit. Find the number of possible passwords.

Solution A password can be constructed by the following sequences:

Step 1 Select a starting letter from the given set.

Step 2 Select a sequence of letter and digits with repetitions.

Step 1 can be performed in $^7C_1$ or seven ways. Since there are 26 letters and 10 digits that can be selected for each of the remaining six characters, and since repetitions are allowed, sequence 2 can be performed in 36^6 or 2,176,782,336 ways. By the multiplication rule, there are 7×2176782336 or 15,237,476,352 different passwords.

The combinations with repetition can also be used to find the number of solutions of certain linear equations in which variables are integers subject to constraints.

Example 2.24 How many solutions are there in $x + y + z + u = 29$ subject to the constraints $x \geq 1$, $y \geq 2$, $z \geq 3$, and $u \geq 0$?

Solution Since

$$x + y + z + u = 29 \tag{2.3}$$

where x, y, z, and u are integers such that $x \geq 1$, $y \geq 2$, $z \geq 3$, and $u \geq 0$, i.e., $x - 1 \geq 0$, $y - 2 \geq 0$, $z - 3 \geq 0$, and $u \geq 0$.

Assume $x_1 = x - 1$, $x_2 = y - 2$, $x_3 = z - 3$. Then,

$$x = x_1 + 1, y = x_2 + 2, z = x_3 + 3 \quad \text{and} \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, u \geq 0$$

From Eq. (2.3),

$$x_1 + 1 + x_2 + 2 + x_3 + 3 + u = 29$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + u = 23$$

Hence, the number of solutions $= {}^{23+4-1}C_{4-1} = {}^{26}C_3 = (26.25.24)/(1.2.3) = 2600.$

EXAMPLE 2.25 How many integral solutions are there to the system of equations

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20 \quad \text{and} \quad x_1 + x_2 = 15$$

where $x_k \geq 0$, $k = 1, 2, 3, 4, 5$.

Solution We have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20 \tag{2.4}$$

and

$$x_1 + x_2 = 15 \tag{2.5}$$

Then from Eqs (2.4) and (2.5),

$$x_3 + x_4 + x_5 = 5 \tag{2.6}$$

$$x_1 + x_2 = 15 \tag{2.7}$$

and given $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$. Then, the number of solutions in Eq. (2.6):

$${}^{5+3-1}C_{3-1} = {}^{7}C_2 = 21$$

and the number of solutions in Eq. (2.7):

$${}^{15+2-1}C_{2-1} = {}^{16}C_1 = 16$$

Hence, the total number of solutions of the given system of equations $= 21 \times 16 = 336.$

2.7 DISCRETE PROBABILITY

The theory of probability has its origin in the games of chance such as gambling and since then it has developed so much that we find its applications in almost all fields of knowledge. In particular, it is useful in solving problems related to mortality and insurance. Even in our daily life, there exists a number of phenomena where we cannot make prediction with certainty or complete reliability.

2.16 COUNTING (COMBINATORIAL) METHOD

Generating functions are useful in solving variety of counting problems. They are generally used to count the number of combinations of various types. One of the varieties of this counting is to count the solutions of the form

$$e_1 + e_2 + e_3 + \cdots + e_n = k$$

where k is a constant and each e_i is a non-negative integer that may be subjected to a specified constraint. After all, generating function will be used to tackle this type of counting problems.

EXAMPLE 2.62 Find the number of non-negative integral solutions to $e_1 + e_2 + e_3 + \cdots + e_n = r$ where $0 \le e_i \le 1$.

Solution Let $G_i(x) = 1 + x$ for each $i = 1, 2, \ldots, r$. Thus, the generating function is $G_1(x)G_2(x)\cdots G_n(x) = (1 + x)^n$ and the number of solutions is ${}^nC_r$.

EXAMPLE 2.63 Find the number of solutions of $e_1 + e_2 + e_3 + e_4 + e_5 = 21$ where e_1, e_2, e_3, e_4, and e_5, are non-negative integers with $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd, and $1 \le e_5 \le 9$.

Solution Let

$$G_1(x) = 1 + x + x^2 + x^3$$
$$G_2(x) = 1 + x + x^2 + x^3$$
$$G_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$
$$G_4(x) = x^2 + x^3 + x^4 + x^5 + x^6$$
$$G_5(x) = x + x^3 + x^5 + x^7 + x^9$$

Thus, the generating function is $G_1(x)\,G_2(x)\,G_3(x)\,G_4(x)\,G_5(x) = (1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$. The number of solutions with the given constraints is the coefficient of x^{21} in the expansion of $(1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$, i.e., we will obtain a term equal to x^{21} by choosing terms from the first two sums x^{e_1} and x^{e_2}, terms from the second two sums x^{e_3} and x^{e_4}, term from the last sum x^{e_5}, so that the exponents, e_1, e_2, e_3, e_4, and e_5 satisfy the equation $e_1 + e_2 + e_3 + e_4 + e_5 = 21$ and the given constraints.

It is observed that the coefficients of x^{21} in the product is 4. Hence, the number of solutions is 4.

EXAMPLE 2.64 Find a generating function for a_r, the number of ways to select r balls from a pile of three green, three white, three blue, and three red balls.

Solution Here, the generating function, will be a multiplication of four factors corresponding to each colour green, white, blue, and red.

Since there are three balls of each colour, each factor will be $(1 + x + x^2 + x^3)$. Hence, the required generating function is $(1 + x + x^2 + x^3)^4$.

EXAMPLE 2.65 Find the generating function for a_r, the number of ways to select r objects from n objects with unlimited repetitions, and also find a_r.

Solution Since each object can be selected unlimitedly, the generating function is then

$$G(x) = (1 + x + x^2 + \cdots)^n = [(1 - x)^{-1}]^n = (1 - x)^{-n}.$$

To find a_r, we shall find the coefficient of x^r in $G(x)$, i.e., coefficient of

$$x^r = (-1)^r \frac{(-n)(-n-1)\cdots(-n+1-r)}{r!}$$

$$= \frac{(n+r-1)!}{r!(n-r)!} = {}^{n+r-1}C_r$$

2.17 PIGEONHOLE PRINCIPLE

The *pigeonhole principle* states that if there are more pigeons than pigeonholes, then there will be at least one pigeonhole with at least two pigeons in it. This principle is applicable to other objects besides pigeons and pigeonholes. Thus, it may be stated that if $(n + 1)$ or more objects are placed into n boxes then there exists at least one box containing two or more objects.

The pigeonhole principle (also known as *Dirichlet drawer* principle or *shoe box* principle) is at times useful in counting principles.

In the *set-theoretic approach*, the pigeonhole principle can be expressed as follows.

Let X and Y be any two finite sets such that $|X| < |Y|$. Then, a function $f: X \rightarrow Y$ cannot be one-to-one, i.e., there exists at least two elements x_1, x_2 in X so that $f(x_1) = f(x_2)$.

For example

(i) Suppose the department of mathematics contains 13 professors. Then 2 of the professors (pigeons) were born in the same month (pigeonholes) out of 12 months.

(ii) Suppose a laundry bag contains many red, white, and blue socks. Then one needs to grab only four socks (pigeons) to be certain of getting a pair with the same colour (pigeonholes).

Corollary 2.2 If n pigeons are assigned to m pigeonholes, then at least one pigeonhole contains two or more pigeons $(m < n)$.

Proof Let m pigeonholes be labelled with the numbers 1 to m, beginning with the pigeon 1, each pigeon is assigned with respect to the pigeonholes with the same number. Since $m < n$, i.e., the number of pigeonholes is less than the number of pigeons, $n - m$ pigeons are left without assigning a pigeonhole. Thus, at least one pigeonhole will be assigned to a second pigeon.

2.17.1 Generalized Pigeonhole Principle

If a pigeonhole is occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or, more pigeons.

Example 2.66 Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution Here, $n = 12$ months are the pigeonholes and $k + 1 = 3$ or $k = 2$. Hence, among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

State-of-art discussion of growth of function is presented in Chapter 8.

1.33 THE INCLUSION–EXCLUSION PRINCIPLE

When the two tasks can be done simultaneously, then both the sum rule and the product rule (will be discussed in detail in Section 2.2) cannot be used. If we add the number of ways to do each task then the ways to do both the tasks are counted twice. Thus, to correct this double counting and to find the number of ways to do one of the tasks, we add the number of ways in which each task can be done and then subtract the number of ways in which both the tasks can be done. This method of counting is called the principle of *inclusion–exclusion*. Sometimes, it is also called the *subtraction* principle.

In *set-theoretic approach* this counting principle can be stated as follows.

Let A and B be two finite sets. To select an element from A there exists $|A|$ ways and $|B|$ ways to select an element from B. The number of ways to select an element from A, or from B, i.e., the number of ways to select an element from their union, is the sum of the number of ways to select an element from A and the number of ways to select an element from B, minus the number of ways to select an element which is both in A and B. Since there are $|A \cup B|$ ways to select an element in either A or B, and $|A \cap B|$ ways to select an element common to both sets, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Considering three sets A, B, and C, the above principle can be formulated as

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

In general, if $A_1, A_2, \ldots, A_n$ are n-finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le 1} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

The following example demonstrates the process of solving a type of counting problem using the above principle.

EXAMPLE 1.72 How many bit strings of length 8 either start with 1 bit or end with 2 bits 00?

Solution Let A be the set containing bit strings of length 8 beginning with 1 bit. We can create a bit string of length 8 that begins with 1 bit, in $2^7 = 128$ ways, i.e., $|A| = 128$. This is possible by the product rule because the first bit can be selected in only one way and each of the other 7 bits can be selected in two ways. Similarly, we can construct a bit string of length 8 ending with 2 bits 00, in $2^6 = 64$ ways, i.e., $|B| = 64$. This is also possible by the product rule, since each of the first 6 bits can be selected in two ways and the last 2 bits can be selected in one way.

Again, the ways to construct a bit string of length 8 starting with 1 bit are the same as the ways to construct a bit string that ends with 2 bits 00. There are $2^5 = 32$ ways to construct such a string, i.e., $|A \cap B| = 32$. This follows by the product rule, because the first bit can be selected in only one way, each of the second through the sixth bits can be chosen in two ways, and the last 2 bits can be selected in one way.

Hence, the number of bit strings of length 8 that begin with 1 bit or end with 2 bits 00, which equals the number of ways to construct a bit string of length 8 that begin with 1 bit or that ends with 2 bits 00, equals

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 128 + 64 - 32 = 160$$

EXAMPLE 1.73 How many positive integers not exceeding 100 are divisible either by 4 or by 6?

Solution The integers which are divisible by 4 are

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100

i.e., there are 25 integers not exceeding 100 which are divisible by 4. Also, the integers divisible by 6 are

6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96

i.e., there exist 16 integers, which are divisible by 6.

Let A be the set possessing the integers which are divisible by 4 and B be the set which contains the integers, divisible by 6, then

$$n(A) = 25, \quad n(B) = 16$$

Thus, the number of integers which are divisible by 4 or 6 are

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 25 + 16 - 8 \qquad \text{[common in both } A \text{ and } B\text{]}$$
$$= 33$$

EXAMPLE 1.74 In a class of discrete mathematics every student possesses a major subject in computer science or mathematics, or both. The number of students having mathematics as a major one (possibly along with mathematics) is 25; the number of students having mathematics as a major one (possibly along with computer science) is 13; and the number of students having both computer science and mathematics as major is 8.

Solution Let A be the set of students in the class majoring in computer science and B be the set of students in the class majoring in mathematics. Then, $A \cap B$ is the set of students in the class who are having both mathematics and computer science as majors. Since every student in the class is majoring in either computer science or mathematics (or both), it follows that number of students in the class is $|A \cup B|$. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$

Hence, there are 30 students in the class. The illustration is shown in Figure 1.42.

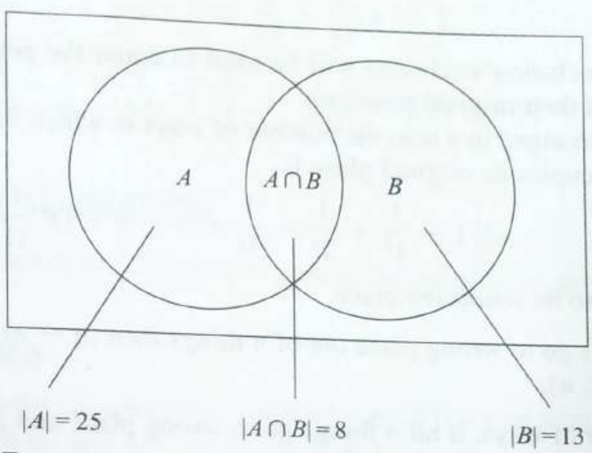

FIGURE 1.42 The set of students in a discrete mathematics class

EXAMPLE 1.75 A computer company receives 40 applications for a job of programmers. Among them 25 knew JAVA, 28 knew ORACLE, and 7 did not know any of the languages. How many of them knew both the languages?

Solution Let J be the set of programmers who knew JAVA and O be the set of programmers who knew ORACLE. Then $J \cup O$ is the set of programmers who knew JAVA or ORACLE (or both) and $J \cap O$ is the set of programmers who knew both the languages. By the principle of inclusion–exclusion, number of programmers knowing both the languages are

$$|J \cap O| = |J| + |O| - |JUO| = 25 + 28 - 33 \quad (\because |J \cup O| = 40 - 7 = 33)$$
$$= 20$$

1.33.1 Applications of Inclusion–Exclusion Principle

Variety of counting problems can be tackled with the use of inclusion–exclusion principle. This principle can be used in *counting* the number of *onto functions* from one finite set to another. This also counts the permutations of objects that leave no object in its original position, which can be termed as *derangements*.

Counting the Number of Onto Functions

The principle of inclusion–exclusion evaluates number of onto functions from a set with m elements to a set with n elements. This evaluation is based on a prepositional form as given below.

Let m and n be positive integers with $m \geq n$. Then there are

$$n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - \cdots + (-1)^{n-1}\,{}^nC_{n-1}\,m$$

onto functions from a set with m elements to a set with n elements. The proof is left to the reader.

An onto function from a set with m elements to a set with n elements corresponds to a way of distribution of the m elements in the domain to n indistinguishable boxes so that no box is empty, and then to associate each of the n elements of the codomain to a box. This implies that the number of onto functions from a set with m elements to a set with n elements is the number of ways to distribute m distinguishable objects to n indistinguishable boxes so that no box is empty multiplied by the number of permutation of a set with n elements.

Derangements

The principle of inclusion–exclusion will be used to count the permutation of n objects that leave no objects in their original positions.

If n things are arranged in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + -1)^n \frac{1}{1!}\right]$$

or, no object goes to its scheduled place.

Note If r things go to wrong place out of n things then $(n-r)$ things go to original place (here $r < n$).

If D_n = number of ways, if all n things go to wrong place and D_r = number of ways, if r things go to wrong place.

If r things go to wrong place out of r, then $(n-r)$ things go to correct places. Then

$$D_n = {}^nC_{n-r}\,D_r$$

If at least p of them are in the wrong places, then

$$D_n = \sum_{r=p}^{n} {}^nC_{n-r}\,D_r$$

where

$$D_r = r!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{r!}\right]$$

EXAMPLE 1.76 A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) at least two of them are in the wrong envelopes and (ii) all the letters are in the wrong envelopes.

Solution

(i) The number of ways in which at least two of them are in the wrong envelopes

$$= \sum_{r=2}^{6} {}^nC_{n-r}\,D_r$$

$$= {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6 \text{ [here } n = 6]$$

$$= {}^6C_4.2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3.3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$
$$+ {}^6C_2.4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
$$+ {}^6C_1.5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$+ {}^6C_0.6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$
$$= 15 + 40 + 135 + 264 + 265 = 719$$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$
$$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right) = 360 - 120 + 30 - 6 + 1 = 265$$