3 Mathematical Logic

LEARNING OBJECTIVES

After reading this chapter you will be able to understand the following:

- The rules of logic that help to understand and reason with statements
- The circumstances under which the formulae will be regarded as being 'true', which is the semantics of the theory
- The compound propositions that play in response to logical equivalences
- The propositional formula that can be expressed in certain normal forms, namely disjunctive and conjunctive involving connectives
- The use of both direct and indirect arguments to derive new results from those already known to be true
- The predicate logic that can be used to express the meaning of statements in mathematics and computer science by the ways in reasoning and exploring relationships between objects
- The necessity of quantification to create a proposition from a propositional function
- The rules of inference that are used to produce valid arguments in propositional logic
- Several methods of proof that are implemented to show the validity of statements

1 INTRODUCTION

Logic is the basis of all mathematical and automated reasoning. In fact, logical reasoning is the essence of mathematics. The *Oxford English Dictionary* defines the word 'logical' as 'correctly reasoned'. It has practical applications in the design of computing machines, in the specification of systems, in artificial intelligence, in computer programming, and in the other areas of computer science.

Mathematical logic is a subarena of mathematics with close interactions with computer science. It includes the study of the expressive power of formal systems and the deductive power of formal proof systems. To understand mathematics, we must think about a correct mathematical argument, i.e., a proof for a theorem. Students of computer science often find it imperative as how important proofs are in computer science. In fact, theorems along with their proofs play essential roles when we verify that computer programs produce the correct output for all possible input values, when we show that algorithm always produces the correct result, and when we create artificial intelligence.

For hardware design, it is a propositional logic which, at present, is most applicable and enables us to handle the desired behaviour of gates and circuits. We discuss the fundamental ideas in Sections 3.2–3.5, and describe the method of truth tables for testing the validity of a formula. In Sections 3.6–3.8, we present the views of semantics (meaning of a propositional statement), namely a formula which is provable in the system if and only if it is always true (a tautology); or, more generally, that it is formally derivable from a set of assumptions if and only if it is true under every assignment of truth values making all the assumptions true. We have then carried on to look on logical equivalence and logical implication in respective Sections 3.9–3.10 of which the former one shows the equivalence of two compound propositions by using truth tables. In Section 3.11, we give a standard form for propositional formulae called *normal forms* which are accepted as a better option to construct truth tables. We introduce arguments in Section 3.12, which hold only compound propositions to be valid. Then we have investigated a collection of rules of inference in propositional logic in Section 3.13, which are among the most important components in producing valid arguments. Next, we set to concentrate on an important way to create a proposition from a propositional function that is quantifications in Section 3.16. It expresses the extent to which a predicate is true over a range of elements. Finally, we introduce the notion of proof, and describe methods and constructive proofs in Section 3.17.

3.2 STATEMENT (PROPOSITIONS)

A statement (or a proposition) is a declarative sentence (i.e., a sentence that declares a fact) which is either true or false but not both; and which is also sufficiently objective, meaningful and precise. The truth or falsity of a statement is called its *truth value*. The truth values 'True' and 'False' of a statement are denoted by T and F, respectively. Also, the value of a statement if true is denoted by 1 and false if expressed by 0.

For example Consider the following sentences:

(i) Kolkata is in India
(ii) $4 + 2 = 6$
(iii) $5 < 7$
(iv) Bangalore is in West Bengal
(v) $x + 2 = 5$
(vi) Where are you going?
(vii) Roses are red
(viii) Go to bed

The sentences (i), (ii), (iii), (iv), and (vii) are statements; among them (iv) is false and others are true. The item (v) is not a proposition (or a statement), since it is neither true nor false. It can be a proposition if we assign a value to the variable x. The item (vi) is a question, not a declarative sentence, hence it is not a statement. Finally, (viii) is not a statement, but a command only.

Statements are usually denoted by the letters $p, q, r, \ldots$. If p denotes the statement 'Bangalore is in West Bengal', then instead of saying that the above statement is false, we can simply represent the value of p as F.

LAWS OF FORMAL LOGIC

Here, we will state two famous laws of formal logic.

1. *Law of contradiction* For every proposition p it is not the same notion that p is both true and false.
2. *Law of intermediate exclusion* If p is a statement (proposition), then either p is true or false, and there is no possibility of intermediate exclusion.

BASIC SET OF LOGICAL OPERATORS/OPERATIONS

In this section, we will discuss three basic logical operators/operations namely, conjunction ($\wedge$), disjunction ($\vee$), and negation ($\sim$) which correspond to the English words like 'and', 'or', and 'not', respectively.

3.4.1 Conjunction (AND, $p \wedge q$)

If any two propositions can be combined by the word 'and', then we can create a new proposition called *compound proposition*. This proposition is actually called the *conjunction* of the original propositions. Symbolically, $p \wedge q$ is read as 'p and q' and represents the conjunction of p and q. Since $p \wedge q$ is a proposition, it has a truth value, and this truth value depends only on the truth values of p and q. Thus, if p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

The truth value of $p \wedge q$ is shown in Table 3.1. Here, the first row says that if p is true and q is true, then $p \wedge q$ is also true. The second row implies that if p is true and q is false, then $p \wedge q$ is false, and so on. It may be noted that there are four rows corresponding to the four possible combinations of T and F for the two propositions p and q.

Table 3.1 $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For example Consider the following four statements:

(i) p: Kolkata is in West Bengal and q: $4 + 4 = 8$
(ii) p: Kolkata is in West Bengal and q: $4 + 4 = 9$
(iii) p: Kolkata is in Orissa and q: $4 + 4 = 8$
(iv) p: Kolkata is in Orissa and q: $4 + 4 = 9$

Here, only the first statement (i) is true. Each of the other statements is false, since at least one of its substatements is false.

Example 3.1 Find the conjunction of the propositions p and q when p is the proposition 'Today is Saturday' and q is the proposition 'It is raining heavily today'.

Solution The conjunction of these propositions, $p \wedge q$, is the proposition 'Today is Saturday and it is raining heavily today'. This proposition is true on rainy Saturday and is false on any day, but not on Saturday, and on Saturday when it does not rain.

3.4.2 Disjunction (OR, $p \vee q$)

If any two propositions can be combined by the word 'or' to form a compound proposition then the proposition is called the *disjunction* of the original propositions. Symbolically, $p \vee q$ is read as 'p or q' and denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q. Thus, if p and q are false, then $p \vee q$ is false, otherwise $p \vee q$ is true.

The truth value of $p \vee q$ is represented in Table 3.2. It may be observed from the table that $p \vee q$ is false in the fourth row when both p and q are false.

TABLE 3.2 $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

For example Consider the following four statements:

(i) p: Kolkata is in West Bengal and q: $4 + 4 = 8$
(ii) p: Kolkata is in West Bengal and q: $4 + 4 = 9$
(iii) p: Kolkata is in Orissa and q: $4 + 4 = 8$
(iv) p: Kolkata is in Orissa and q: $4 + 4 = 9$

Here, only the last statement (iv) is false. Each of the other statements is true, since at least one of its substatements is true.

EXAMPLE 3.2 Find the disjunction of the propositions p and q where p is the proposition 'Today is Saturday' and q is the proposition 'It is raining heavily today'.

Solution The disjunction of p and q, $p \vee q$, is the proposition 'Today is Saturday or it is raining heavily today'.

The English word 'or' can be used in two different notions, namely, as an *inclusive* ('and/or') or *exclusive* ('either/or').

For example Consider the following two statements:

(i) p: Ashish will go to Kolkata or to Bangalore.
(ii) q: There is something wrong with the fan or switch.

In the statement (i), the disjunction of the statement p is used in exclusive sense (p or q but not both); i.e., one or other possibility exists but not both.

In the statement (ii), the disjunctive is implemented in an inclusive sense (p or q or both). Here, at least one of the two possibilities will occur. However, both could have occurred. In the inclusive sense, we shall always use 'or' unless it is stated.

EXAMPLE 3.3 Let p be 'Rekha speaks Bengali' and let q be 'Rekha speaks Oriya'. Give a simple verbal sentence which describes each of the following:
(i) $p \vee q$ (ii) $p \wedge q$

Solution

(i) Rekha speaks Bengali or Oriya.
(ii) Rekha speaks Bengali and Oriya.

Example 3.4 Assign a truth value to each of the following statements:
(i) $6 + 4 = 10 \vee 0 > 2$
(ii) $5 \times 4 = 21 \vee 9 + 7 = 17$

Solution

(i) True, since one of its components, i.e., $6 + 4 = 10$ is true.
(ii) False, since both of its components are false.

3.4.3 Negation (NOT, $\sim p$)

Given any proposition p, a new proposition, called the *negation* of p, can be formed by stating 'It is not the case that . . .' or 'It is false that . . .' for p, or, if possible by inserting in p the word 'not'. Symbolically, $\sim p$ is read as 'not p', and denotes the negation of p. Thus, if p is true, then $\sim p$ is false; and if p is false, then $\sim p$ is true. The truth value of $\sim p$ is represented in Table 3.3.

TABLE 3.3 $\sim p$

p	$\sim p$
T	F
F	T

For example Consider the following six statements:
(i) Kolkata is in India
(ii) It is not the case that Kolkata is in India
(iii) Kolkata is not in India
(iv) $4 + 4 = 9$
(v) It is not the case that $4 + 4 = 9$
(vi) $4 + 4 \neq 9$

Here, (ii) and (iii) are each the negation of (i); and (v) and (vi) are each the negation of (iv). Since (i) is true, (ii) and (iii) are false; and since (iv) is false, (v) and (vi) are true.

EXAMPLE 3.5 Find the negation of the following propositions:
(i) Today is Saturday.
(ii) It is a rainy day.
(iii) If it snows, Mona does not drive the car.

Solution
(i) Today is not Saturday.
(ii) It is not a rainy day.
(iii) It snows and Mona drives the car.

EXAMPLE 3.6 Let p: Priya is tall and q: Priya is beautiful. Write the following statements in symbolic form.
(i) Priya is tall and beautiful.
(ii) Priya is tall but beautiful.
(iii) It is false that Priya is short or beautiful.
(iv) Priya is tall or Priya is short and beautiful.

Solution
(i) $p \wedge q$ (ii) $p \wedge \sim q$ (iii) $\sim(\sim p \vee q)$ (iv) $p \vee (\sim p \wedge q)$

3.5 PROPOSITIONS AND TRUTH TABLES

Let $P(p, q)$ denote an expression constructed from logical variables $p, q, \ldots$, which take on the value TRUE (T) or FALSE (F), and which operate on the logical connectives $\wedge, \vee, \sim$ (and others discussed subsequently). Such an expression is called a *proposition*.

One of the important characteristics of a proposition $P(p, q)$ is that its truth value de- upon the truth value of its variables, i.e., we can determine the truth value of a propositi we know the truth values of each of its variables. We will construct this relationship thro truth table.

For example Consider the proposition $\sim(p \vee q)$. Table 3.4 shows how the truth ta $\sim(p \vee q)$ is constructed. First of all, columns are to be marked by p, q, $p \vee q$, $\sim(p \wedge q)$. Then fill up the p and q columns with the logically possible combinations of T's and F's and subsequently the $(p \vee q)$ and $\sim(p \vee q)$ columns with the appropriate tr values. Hence, the truth table, $\sim(p \vee q)$, is shown in Table 3.5.

TABLE 3.4 Construction of $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

TABLE 3.5 $\sim(p \vee q)$

p	q	$\sim(p \vee q)$
T	T	F
T	F	F
F	T	F
F	F	T

3.5.1 Connectives

Statements can be connected by words like 'not', 'and', etc. These words are known as *logical connectives*. The statements which do not contain any of the connectives are called *atomic statements* or *simple statements* (or, *primitive statement*).

The common connectives used are negation (~), and ($\wedge$), or ($\vee$), if … then ($\rightarrow$ or $\Rightarrow$), if and only if ($\leftrightarrow$ or $\Leftrightarrow$), equivalence ($\equiv$). We will use these connections along with symbols to combine various simple statements.

EXAMPLE 3.7 Write the following statements in symbolic form:

(i) If Avinash is not in a good mood or he is not busy, then he will go to Kharagpur.

(ii) If Sayantan knows object-oriented programming and oracle, then he will get a job.

Solution

(i) Let p: Avinash is in a good mood, q: Avinash is busy and r: Avinash will go to Kharagpur, then $(\sim p \vee \sim q) \rightarrow r$.

(ii) Let p: Sayantan know object-oriented programming, q: Sayantan know oracle, r: Sayantan will get a job, then $(p \wedge q) \rightarrow r$.

EXAMPLE 3.8 Let p: Babu is rich, q: Babu is happy.

Give a simple verbal sentence which describes each of the following statements:

(i) $p \vee q$ (ii) $p \wedge q$ (iii) $q \rightarrow p$ (iv) $p \vee \sim q$
(v) $q \leftrightarrow \sim p$ (vi) $\sim p \rightarrow q$ (vii) $\sim \sim p$ (viii) $(\sim p \wedge q) \rightarrow p$

Solution The meaning of the symbols $\wedge$, $\vee$, ~, $\rightarrow$, and $\leftrightarrow$ are 'and', 'or', 'it is false' 'if … then', and 'if and only if', respectively. Then the above statements are expressed as

(i) Babu is rich or Babu is happy.
(ii) Babu is rich and Babu is happy.
(iii) Babu is happy then Babu is rich.
(iv) Babu is rich or Babu is not happy.
(v) Babu is happy if and only if Babu is not rich.
(vi) Babu is not rich then Babu is happy.
(vii) It is not true that Babu is not rich.
(viii) If Babu is not rich and happy then Babu is rich.

3.5.2 Compound Propositions

Many statements (or propositions) are composite that is composed of subpropositions by means of logical operators or connections. Such statements are referred to *compound* (or *composite*) statements. The fundamental property of a compound statement is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are combined to construct the compound statement.

For example

(i) 'Sohan is intelligent or studies every night' is a compound proposition with subpropositions 'Sohan is intelligent' and 'Sohan studies every night'.
(ii) 'The sun is shining and the sky is blue' is a compound proposition with subpropositions 'The sun is shining' and 'the sky is blue'.

Example 3.9 Construct a truth table for each of the following compound propositions:
(i) $(p\wedge q)\vee(p\wedge r)$ (ii) $\sim(p\vee q)\vee(\sim p\wedge\sim q)$

Solution Tables 3.6 and 3.7

(i) **Table 3.6** $(p\wedge q)\vee(p\wedge r)$

p	q	r	$p\wedge q$	$p\wedge r$	$(p\wedge q)\vee(p\wedge r)$
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

(ii) **Table 3.7** $\sim(p\vee q)\vee(\sim p\wedge\sim q)$

p	q	$\sim p$	$\sim q$	$p\vee q$	$\sim(p\vee q)$	$(\sim p\wedge\sim q)$	$\sim(p\vee q)\vee(\sim p\wedge\sim q)$
F	F	T	T	F	T	T	T
F	T	T	F	T	F	F	F
T	F	F	T	T	F	F	F
T	T	F	F	T	F	F	F

3.6 ALGEBRA OF PROPOSITIONS

Propositions that satisfy various laws are listed in Table 3.12. They are similar to algebra and useful to simplify expression. In fact, the connective $\vee$ is often treated like +, and connective $\wedge$ is often dealt like •.

TABLE 3.12 Laws of the algebra of propositions

1. (a) $p \vee p \equiv p$	**Idempotent laws**	(b) $p \wedge p \equiv p$
2. (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	**Associative laws**	(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3. (a) $p \vee q \equiv q \vee p$	**Commutative laws**	(b) $p \wedge q \equiv q \wedge p$
4. (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	**Distributive laws**	(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. (a) $p \vee T \equiv T$	**Identity laws**	(b) $p \wedge T \equiv p$
6. (a) $p \vee F \equiv p$		(b) $p \wedge F \equiv F$
7. (a) $p \vee \sim p \equiv T$	**Complement laws**	(b) $p \wedge \sim p \equiv F$
8. (a) $\sim T \equiv F$		(b) $\sim F \equiv T$
9. $\sim\sim p \equiv p$	**Involution law**	
10. (a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$	**De Morgan's laws**	(b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

3.7 PROPOSITIONAL FUNCTIONS

Let A be a given set. A *propositional function* (or, an *open sentence* or *condition*) defined on A is an expression $P(x)$ which has the property that $P(a)$ is true or false for each $a \in A$. Then, $P(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $P(x)$, and the set T_P of all elements of A for which $P(a)$ is true, is called the *truth set* of $P(x)$. In other words,

$$T_P = \{x : x \in A, P(x) \text{ is true}\} \quad \text{or} \quad T_P = \{x : P(x)\}$$

When A is some set of numbers, then $P(x)$ takes the form of an equation or inequality involving the variable x.

EXAMPLE 3.14 Let $P(x)$ denote the sentence '$x + 4 > 3$'. Investigate $P(x)$ as a propositional function on each of the following sets:

(i) N, the set of natural numbers
(ii) $M = \{-1, -2, -3, \ldots\}$
(iii) $C = \{$the set of complex numbers$\}$

Solution

(i) Yes, $P(x)$ is a propositional function on N.
(ii) Although $P(x)$ is false for every element in M, yet $P(x)$ is still a propositional function on M.
(iii) No, since $i + 4 > 3$ (where $i = \sqrt{-1}$ is an imaginary quantity) does not have any meaning. In other words, inequalities are not defined for complex numbers.

EXAMPLE 3.15 Find the truth set of each of the following propositional function $P(x)$ defined on the set N of positive integers:

(i) $P(x) : x + 3 < 7$ (ii) $P(x) : x + 5 > 8$ (iii) $P(x) : x + 4 < 1$

Solution

(i) The truth set of $P(x)$ is

$P(x) = \{x : x \in N, x + 3 < 7\} = \{1, 2, 3\}$

consisting of integers less than 4.

(ii) The truth set of $P(x)$ is

$P(x) = \{x : x \in N, x + 5 > 8\} = \{4, 5, 6, \ldots\}$

consisting of all integers greater than 3.

(iii) The truth set of $P(x)$ is

$P(x) = \{x : x \in N, x + 4 < 1\} = \phi$

the null set. In other words, $P(x)$ is not true for any positive integer in N.

8 TAUTOLOGIES AND CONTRADICTIONS

In some proposition $P(p, q)$, if the last column of their truth tables contain only T, i.e., the propositions are true for any truth values of their variables, then such propositions are called *tautologies*. On the other hand, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of its truth table, i.e., if it is false for any truth values of its variables. Furthermore, a proposition that is neither a tautology nor a contradiction is called a *contingency*. So, the proposition 'p or not p', i.e., $p \vee \sim p$, is a tautology, and the proposition 'p and not p', i.e., $p \wedge \sim p$ is a contradiction. This can be verified by their truth tables, as shown in Tables 3.13 and 3.14. Here, the truth tables possess only two rows since each proposition has only one variable p.

TABLE 3.13 $p \vee \sim p$

p	$\sim p$	$(p \vee \sim p)$
T	F	T
F	T	T

TABLE 3.14 $p \wedge \sim p$

p	$\sim p$	$(p \wedge \sim p)$
T	F	F
F	T	F

Note The negation of a tautology is a contradiction, since it is always false. Again, the negation of a contradiction is a tautology, since it is always true.

EXAMPLE 3.16 Verify that the proposition $p \vee \sim(p \wedge q)$ is a tautology.

Solution

TABLE 3.15 Illustration of a tautology

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \vee \sim(p \wedge q)$
F	F	F	T	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

Since, in Table 3.15, the truth value of $p \vee \sim(p \wedge q)$ is true for all values of p and q the proposition is a tautology.

3.8.1 System Specifications (Consistency)

Translating sentences in natural language (such as English) into logical expressions is an es-tial part of specifying both hardware and software systems. System and software engi-consider natural language and produce precise and unambiguous specifications that can be used as the basis for system development. The following example illustrates how comp-propositions can be used in this process.

Example 3.17 Translate the specification 'The automated reply cannot be sent when the system is full' using logical connectives.

Solution Let p: The automated reply can be sent and q: The file system is full. Then, $\sim p$: It is not the case that the automated reply can be sent. Again, $\sim p$: The automated reply cannot be sent. Consequently, the specification can be represented by the conditional statement $q \rightarrow$

Example 3.18 Investigate the following system specifications as consistent one:

'The diagnostic message is stored in the buffer or it is retransmitted'
'The diagnostic message is not stored in the buffer'
'If the diagnostic message is stored in the buffer, then it is retransmitted'

Solution Let p: The diagnostic message is stored in the buffer and q: The diagnostic message is retransmitted. The specifications can then be written as $p \vee q$, $\sim p$, and $p \rightarrow q$. All the three specifications will be true if p would have been false, so that $\sim p$ is true. To make $p \vee q$ true p must be made false and q is true. Finally, because $p \rightarrow q$ is true when p is false and q is true we conclude that the given specifications are consistent.

3.8.2 Principle of Substitution

We have already discussed the construction of tautology. However, there exists another procedure of creating tautology through the principle of substitution (or replacement process), which we will describe now.

The *principle of substitution* illustrates a substitution (or replacement) process of a formula F by another formula G, i.e. if F can be obtained from G by substituting (or replacing) formulae for some variables of G following the condition that the same formula is replaced for the same variables each time, if needed. From logical point of view, we demonstrate the above definition as follows.

Assume a formula $F:P \rightarrow (q \rightarrow r)$. If we now replace $(q \rightarrow r)$ by an equivalent formula $\sim q \vee r$ in F, we get another formula $G:p \pm (\sim q \vee r)$. Then, we can verify that formulae F and G are equivalent to each other. This process of obtaining G from F is known as the *substitution principle*.

Note Actually, to construct a substitution instance of a formula, substitutions are made for the simple proposition (without connectives) and not for the compound proposition. For instance, $p \rightarrow q$ is not a substitution instance of $p \rightarrow \sim r$, because it must be replaced by r but not by $\sim r$.

Example 3.19 Show that the proposition $(p \wedge \sim q) \vee \sim (p \wedge q)$ is a tautology.

Solution The given proposition can be expressed in the form $p \wedge \sim p$, where $p = p \wedge \sim q$. Since $p \wedge \sim p$ is a tautology, then by the principle of substitution, $(p \wedge \sim q) \vee \sim (p \wedge q)$ is also a tautology.

Example 3.20 Prove that $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\sim q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$.

Solution It is known that $q \rightarrow r \Leftrightarrow \sim q \vee r$. Now, replacing $q \rightarrow r$ by $\sim q \vee r$, we get $p \rightarrow (\sim q \vee r)$, which is equivalent to $\sim p \vee (\sim q \vee r)$. Then,

$$\sim p \vee (\sim q \vee r) \Leftrightarrow (\sim p \vee \sim q) \vee r \Leftrightarrow \sim (p \wedge q) \vee r \Leftrightarrow (p \wedge q) \rightarrow r$$

by associativity of $\vee$, De Morgan's law etc.

Note It is possible to substitute more than one variable by other variables provided all the substitutions are assumed to be occurred simultaneously.

Example 3.21 Show that $(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (p \vee r) \rightarrow q$.

Solution It is known that $(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (\sim p \vee q) \wedge (\sim r \vee q)$. Then, replacing $p \rightarrow q$ and $r \rightarrow q$ by $(\sim p \vee q)$ and $(\sim r \vee q)$, respectively, we get

$$(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (\sim p \wedge \sim r) \vee q \qquad [\because (A_1 \vee A_2) \wedge (A_3 \vee A_2) \Leftrightarrow (A_1 \vee A_3) \vee A_2]$$

$$\Leftrightarrow \sim (p \vee r) \vee q \qquad [\text{replacing } (\sim p \wedge \sim r) \text{ by } \sim (p \vee r)]$$

$$\Leftrightarrow (p \vee r) \rightarrow q \qquad [\because \sim p \vee q \Leftrightarrow (p \rightarrow q)]$$

3.9 LOGICAL EQUIVALENCE

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by $P(p, q, \ldots) \equiv Q(p, q, \ldots)$, if they have the identical truth tables. The notion can also be defined as the propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are logically equivalent if $P \leftrightarrow Q$ is a tautology. The equivalence of P and Q is also denoted by $P \Leftrightarrow Q$.

For example

(i) Consider the truth values of $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ as shown in Tables 3.16 and 3.17. Here, it is seen that both the truth values are the same, i.e., both the propositions are false in the first row and true in the other three rows. Accordingly, we can write $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$. In other words, the propositions are logically equivalent.

(ii) Consider the statement 'It is not the case that Kolkata is in West Bengal and 4 + 4 = 9'. This statement can be written in the form $\sim(p \wedge q)$, where p: Kolkata is in West Bengal and q: $4 + 4 = 9$. However, $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$. Thus, the statement 'Kolkata is not in West Bengal, or 4 + 4 is not equal to 9' carries the same meaning as the given statement, i.e., the propositions are logically equivalent, as shown in Tables 3.16 and 3.17.

Table 3.16 $\sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Table 3.17 $(\sim p) \vee (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3.9.1 De Morgan's Laws

We generally use a truth table to show the equivalency of two compound propositions. In particular, the compound propositions p and q are equivalent if and only if the columns in the truth tables agree. There are two laws of logical equivalences, known as *De Morgan's laws*. The laws are

I. $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
II. $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

Actually, the first law computes the negation of a disjunction by taking the conjunction of the negations of the compound propositions. This has been depicted in Table 3.18. Similarly, in the second one, the negation of a conjunction is formed by taking the disjunction of the negation of the component propositions. The usage of De Morgan's laws is shown in the following example.

Example 3.22 By using De Morgan's laws, express the negation of
'Rina has a cellphone and she has a laptop computer'
and
'Papu will go to the concert or Babu will go to the concert'

Solution Let p: Rina has a cellphone and q: Rina has a laptop computer. Then, 'Rina has a cellphone and she has a laptop computer' can be represented by $p \wedge q$. By De Morgan's laws, $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$. Consequently, the negation of the original statement can be expressed as 'Rina does not have a cellphone and she does not have a laptop computer'.

Let r: Papu will go to the concert and s: Babu will go to the concert. Then, 'Papu will go to the concert or Babu will go to the concert' can be represented by $r \vee s$. By De Morgan's laws, $\sim(r \vee s) \equiv (\sim r) \wedge (\sim s)$. Consequently, 'Papu will not go to the concert or Babu will not go to the concert'.

Example 3.23 By using truth table, show that $\sim(p \vee q)$ is equivalent to $(\sim p) \wedge (\sim q)$.

Solution

Table 3.18 Equivalence of $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

It may be noticed from Table 3.18 that the truth tables in the two columns headed by $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ are identical.

EXAMPLE 3.24 By using Table 3.19, show that $p \wedge (q \vee r)$ is equivalent to $(p \wedge q) \vee (p \wedge r)$.

Solution

TABLE 3.19 Equivalence of $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the two columns, headed by $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$, of the truth Table 3.19 agree, the two compound propositions are identical. This shows that $p \wedge (q \vee r)$ is equivalent to $(p \wedge q) \vee (p \wedge r)$.

EXAMPLE 3.25 Show that $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are equivalent.

Solution Table 3.20

TABLE 3.20 Equivalence of $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
F	F	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
T	T	T	T	T	T

EXAMPLE 3.26 Among the two restaurants next to each other, one has a sign that says 'Good food is not cheap' and the other has a sign that 'Cheap food is not good'. Investigate the signs regarding their equivalence.

Solution Let p: Food is good and q: Food is cheap. The first sign says $p \rightarrow \sim q$ and the second one says $q \rightarrow \sim p$.

TABLE 3.21 Equivalence of $p \rightarrow \sim q$ and $q \rightarrow \sim p$

p	q	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$q \rightarrow \sim p$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	T	F	F	F	F

From the truth table shown in Table 3.21, it is observed that both the signs are equivalent.

3.10 LOGICAL IMPLICATION

A proposition $P(p, q, \ldots)$ is said to *logically imply* a proposition $Q(p, q, \ldots)$, written $P(p, q, \ldots) \Rightarrow Q(p, q, \ldots)$. If $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true. The notion can also be defined as: A proposition P is said to logically imply a proposition Q if $P \Rightarrow Q$ is a tautology.

For example Consider Table 3.22. It is observed that p is true in rows 1 and 2, and consequently, $p \vee q$ is also true in these cases. Thus, $p \Rightarrow p \vee q$. Now, if $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true, then the argument $P(p, q, \ldots) \vdash Q(p, q, \ldots)$ is valid and converse. Moreover, the argument $P \vdash Q$ is valid if and only if the conditional statement $P \Rightarrow Q$ is always true, i.e., a tautology.

TABLE 3.22 $p \Rightarrow p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 3.27 Show that $p \leftrightarrow q$ logically implies $p \Rightarrow q$.

Solution First of all the tables of $p \leftrightarrow q$ and $p \Rightarrow q$ are constructed as shown in Table 3.23. From the table, it is observed that $p \leftrightarrow q$ is true in rows 1 and 4 and $p \Rightarrow q$ is also true in these cases.

TABLE 3.23 $(p \leftrightarrow q) \rightarrow (p \Rightarrow q)$

p	q	$p \leftrightarrow q$	$p \Rightarrow q$
F	F	T	T
F	T	F	T
T	F	F	F
T	T	T	T

EXAMPLE 3.28 Show that $p \wedge q$ logically implies $p \leftrightarrow q$.

Solution Consider the truth tables of $p \wedge q$ and $p \leftrightarrow q$ as shown in Table 3.24. From the table, it is seen that $p \wedge q$ is true only in row 1 and simultaneously the proposition $p \leftrightarrow q$ is also true in this case. Thus, $p \wedge q$ logically implies $p \leftrightarrow q$.

TABLE 3.24 $(p \wedge q) \rightarrow (p \leftrightarrow q)$

p	q	$p \wedge q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

3.11 NORMAL FORMS

One can examine the equivalency of logical expressions P and Q using the truth table. However, the process is tedious to tackle, when the number of variables increases. Convenient option lies in the transformation of the expressions P and Q to standard forms of expressions P^* and Q^*, and then a comparison of P^* and Q^* shows whether $P \equiv Q$. These standard forms are known as *normal forms* or *canonical forms*.

For simplicity we use the words *product* in place of conjunction and *sum* in place of disjunction. Some of the basic normal forms are as follows:

I. Disjunctive normal form (dnf)
II. Conjunctive normal form (cnf)

3.11.1 Disjunctive Normal Form

In a logical expression, a product of the variables and their negation is called an *elementary product*. For instance, $P \wedge \sim R$, $Q \wedge P \wedge \sim R$, etc. are elementary products. Also, the sum of the variables and their negations is called an *elementary sum*. For instance, $P \vee \sim Q \sim P \vee \sim Q \vee \sim R$ are elementary sums.

The elementary sums or products satisfy the following properties:

(i) An elementary sum is identically *true* if and only if it contains at least one pair of factors in which one is the negation of the other. (A part of the elementary sum of product which is itself an elementary sum of product is called a *factor* of the original sum or product.)

(ii) An elementary product is identically *false* if and only if it contains at least one pair of factors in which one is negation of the other.

A logical expression is called a *disjunctive normal form*, abbreviated as *dnf*, if it is a sum of elementary products.

For example The forms $p \vee (q \wedge r)$ and $p \vee (\sim q \wedge r)$ are in dnf, but $p \wedge (q \vee r)$ is not in dnf. For example, consider the 'exclusive OR' $P \oplus Q$, as defined in Table 3.25. That is, $P \oplus Q$ is true iff one of P or Q is true but not both. Thus, $P \oplus Q$ is equivalent to the form $(\sim P \wedge Q) \vee (P \wedge \sim Q)$ which is its dnf.

TABLE 3.25 Exclusive OR/$P \oplus Q$

P	Q	$P \oplus Q$
F	F	F
F	T	T
T	F	T
T	T	F

In general, we obtain the dnf for an n-variable propositional form $f(P_1, P_2, \ldots, P_n)$ from its truth table as follows.

For each row in which $f(P_1, P_2, \ldots, P_n)$ assumes the value T, we form the conjunction $P_1 \wedge P_2 \wedge \cdots \wedge \sim P_k \cdots \wedge \ldots \sim P_n$, where we take P_k if there is a T in the kth position in the row and $\sim P_k$ if there is an F there. This conjunction is called a *minterm*. Then we form the disjunction of the minterms as

$$(P_1 \wedge \sim P_2 \wedge \cdots P_n) \vee (\sim P_1 \wedge P_2 \wedge \cdots \sim P_n) \vee \cdots \vee (P_1 \wedge P_2 \wedge \cdots P_n)$$

Thus, the disjunction of the minterms yields the dnf.

EXAMPLE 3.29 Find the dnf for the propositional form $f(P, Q, R)$ defined as (see Table 3.26).

TABLE 3.26 dnf of $f(P, Q, R)$

P	Q	R	$f(P, Q, R)$
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

Solution The dnf is expressed as $(\sim P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (P \wedge Q \wedge R)$, which is the required dnf.

EXAMPLE 3.30 Find the dnf of $\sim(P \vee Q) \Leftrightarrow (P \wedge Q)$.

Solution

$$\sim(P \vee Q) \Leftrightarrow (P \wedge Q) = (\sim(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \sim(P \wedge Q))$$

[Since $R \Leftrightarrow S = (R \wedge S) \vee (\sim R \wedge \sim S)$]

$$= (\sim P \wedge \sim Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\sim P \vee \sim Q))$$
$$= (\sim P \wedge \sim Q \wedge P \wedge Q) \vee (P \vee Q \wedge \sim P) \vee (P \vee Q \wedge \sim Q)$$
$$= (\sim P \wedge \sim Q \wedge P \wedge Q) \vee (P \wedge \sim P) \vee (Q \wedge \sim P) \vee (P \wedge \sim Q) \vee (Q \wedge \sim Q)$$

EXAMPLE 3.31 Determine the dnf of $p \Rightarrow ((p \Rightarrow q) \wedge \sim(\sim q \vee \sim p))$.

Solution

$$p \Rightarrow ((p \Rightarrow q) \wedge \sim(\sim q \vee \sim p))$$
$$\equiv \sim p \vee ((p \Rightarrow q) \wedge \sim(\sim q \vee \sim p)) \equiv \sim p \vee ((\sim p \vee q) \wedge \sim(\sim q \vee \sim p))$$
$$\equiv \sim p \vee ((\sim p \vee q) \wedge (q \wedge p)) \equiv \sim p \vee ((\sim p \wedge (q \wedge p)) \vee (q \wedge (q \wedge p))$$
$$\equiv \sim p \vee ((\sim p \wedge q) \wedge p) \vee ((q \wedge q) \wedge p) \equiv \sim p \vee (F) \vee (p \wedge q)$$
$$\equiv \sim p \vee (p \wedge q)$$

which is the required dnf.

3.11.2 Conjunctive Normal Form

If a form is a product of elementary sums then that form is called a *conjunctive normal form*. It is abbreviated as *cnf*.

For example The forms $p \wedge r$ and $\sim p \wedge (q \vee r)$ are in cnf.

EXAMPLE 3.32 Find the cnf of the following:

(i) $p \wedge (p \Rightarrow q)$
(ii) $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q)$

Solution

(i) $p \wedge (p \Rightarrow q) \equiv p \wedge (\sim p \vee q)$ (which is in the cnf)

(ii) $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q) \equiv (q \vee (p \wedge r)) \wedge (\sim(p \vee r) \wedge \sim q)$

$$\equiv q \vee (p \wedge r) \vee (\sim p \wedge \sim r) \vee \sim q$$
$$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$$

(which is the required cnf)

3.12 ARGUMENTS

An *argument* is a positive declaration (or an assertion) that a given set of propositions $P_1, P_2, \ldots, P_n$ called *premises*, yields another proposition Q, called the *conclusion*. Such an argument can be denoted by either of its *tautological form*, i.e., $P_1, P_2, \ldots, P_n \vdash Q$ (here, '$\vdash$' means then) or,

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow Q$$

Thus, an argument

$$P_1, P_2, \dots, P_n \vdash Q$$

is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \dots, P_n$ are true; or, in other way, the propositions $P_1, P_2, \dots, P_n$ together with another proposition Q will be a *valid argument* if

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow Q$$

is a tautology. An argument which is not valid is called a *fallacy*.

For example

(i) The argument represented as $p, p \Rightarrow q \vdash q$ is a valid one. Here, the rule is called the *law of detachment*. The proof of this law can be readily available from Table 3.27. In particular, both p and $p \Rightarrow q$ are true in row 1, and in this case q is also true.

(ii) The following argument is a fallacy: $p \Rightarrow q \vdash q$ because $p \Rightarrow q$ and q, both are true in row 3 in Table 3.27. However, in this case p is false.

Thus, the argument $P_1, P_2, \dots, P_n \vdash Q$ is valid if and only if the proposition $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow Q$ is a tautology.

TABLE 3.27 The law of detachment

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A *fundamental principle of logical reasoning* states that 'if p implies q and q implies r then p implies r'; i.e., the following argument $p \Rightarrow q,\ \ q \Rightarrow r \vdash p \Rightarrow r$ is valid. This rule is known as *law of syllogism*. The rule will be verified in the next example.

EXAMPLE 3.33 Verify the law of syllogism by a truth table, i.e., show that the proposition $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Solution The truth table of the law is given below. It can be seen from Table 3.28 that the premises $p \Rightarrow q$ and q are true in rows numbered 1, 5, 7, 8. Also, the conclusion $p \Rightarrow r$ is true in the said rows. So, the argument is valid. It may be observed from the table that, since there exists three variables p, q, and r, the truth table requires $2^3 = 8$ rows.

TABLE 3.28 Verification of law of syllogism

p	q	r	$[(p$	$\to$	$q)$	$\wedge$	$(q$	$\to$	$r)]$	$\to$	$(p$	$\to$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
	Step		1	2	1	3	1	2	1	4	1	2	1

Now, we use the above theory to arguments involving statements. We make stress on the validity of an argument which neither depends upon the truth values nor upon the content of the statements appearing in the argument, but upon the particular form of the argument. This is illustrated in the following example.

Example 3.34 Consider the following argument:

p: If a person is illiterate, he is unhappy

q: If a person is unhappy, he dies young

... ...

r: Illiterate persons die young

Investigate the validity of the argument.

Solution Here, the statements p and q denote the premises and the statement r represents the conclusion of the argument. The given argument is of the form

$$p \Rightarrow q, \quad q \Rightarrow r \vdash p \Rightarrow r$$

Hence, by Example 3.33 the argument (law of syllogism) is valid.

Example 3.35 Show that the following argument is valid:

$$p \to \sim q, r \to q, r \vdash \sim p$$

Solution First of all, the truth tables of the premises and the conclusion are constructed, which is shown in Table 3.29. It is observed from the table that $p \to \sim q$, $r \to q$, and r are true simultaneously in the fifth row, in which $\sim p$ is also true. Thus, the argument is valid.

Table 3.29 Validation of the argument $p \to \sim q, r \to q, r \vdash \sim p$

p	q	r	$p \to \sim q$	$r \to q$	$\sim q$
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

Example 3.36 Show that the following argument is a fallacy:

$$p \to q, \ \sim p \vdash \ \sim q$$

Solution The truth table of $[(p \to q) \wedge \ \sim p] \to \sim q$ is first constructed (Table 3.30). Since the proposition $[(p \to q) \wedge \ \sim p] \to \ \sim q$ is not a tautology, the argument is a fallacy. In the same fashion, the argument is a fallacy since in the third line of the truth table $p \to q$ and $\sim p$ are true, but $\sim q$ is false.

Table 3.30 Illustration of the fallacy of an argument $p \to q, \ \sim p \vdash \ \sim q$

p	q	$p \to q$	$\sim p$	$(p \to q) \wedge \sim p$	$\sim q$	$[(p \to q) \wedge \sim p] \to \sim q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

3.13 RULES OF INFERENCE

It is observed in logical reasoning that a certain number of propositions are assumed to be true, and based on that assumption some other propositions are derived or inferred. The propositions that are assumed to be true are called *premises* or *hypothesis* and the proposition derived by using the rules of inference is called a *valid argument*; in other words, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. *Rules of inference* can be used to derive new statements from the existing statements. These are the basic tools for constructing valid arguments and establishing the truth of sentences.

We will present here some of the important rules of inference, as shown in Table 3.31.

TABLE 3.31 Several rules of inference

Rules of inference	Tautological form	Name
$\dfrac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\dfrac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
$\dfrac{\begin{matrix} p \\ q \end{matrix}}{\therefore p \wedge q}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\dfrac{\begin{matrix} p \\ p \rightarrow q \end{matrix}}{\therefore q}$	$[(p \rightarrow q) \wedge p] \rightarrow q$	Modus Ponens
$\dfrac{\begin{matrix} p \rightarrow q \\ \sim q \end{matrix}}{\therefore \sim p}$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus Tollens
$\dfrac{\begin{matrix} p \rightarrow q \\ q \rightarrow r \end{matrix}}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$\dfrac{\begin{matrix} p \vee q \\ \sim p \end{matrix}}{\therefore q}$	$[(p \vee q) \wedge \sim p] \rightarrow q$	Disjunction Syllogism

Now, we will discuss some of the important rules of inference in propositional logic, considering those statements, which are logically correct arguments.

3.13.1 Law of Detachment (or Modus Pones)

Here, the tautology is $(p \wedge (p \rightarrow q)) \rightarrow q$, which is the basis of the rule of inference called the *law of detachment*, or *modus pones* (modus pones is a *Latin word* for *mode that affirms*). This tautology is presented in the following valid argument form

$$\dfrac{\begin{matrix} p \\ p \rightarrow q \end{matrix}}{\therefore q}$$

The assertions above the horizontal line are called *premises* (or *hypothesis*). The assertion below the line is called the *conclusion* (the symbol '∴' denotes 'therefore'). In particular, modus pones expresses that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true.

For example Assume that the conditional statement, 'If Tanu gets a first class', then 'Tanu will get a job' and its hypothesis, 'Tanu gets a first class', are true. Then from modus pones, it follows that the conclusion of the conditional statement, 'Tanu will get a job', is true.

Let p: Tanu gets a first class and q: Tanu will get a job. Then the premises are p and $p \to q$, the conclusion is q. The inferential form is thus

$$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$$

It may also be noted that if one or more of the valid argument's premises is false, then the valid argument can lead to an incorrect solution.

Example 3.37 Determine whether the argument given here is valid and also determine whether its conclusion must be true because of the validity of the argument,

$$\text{'If } \sqrt{3} > \frac{7}{4}, \text{ then } (\sqrt{3})^2 > \left(\frac{7}{4}\right)^2. \text{ Consequently, } 3 > \frac{49}{16}\text{'.}$$

Solution Let

$$p : \sqrt{3} > \frac{7}{4}$$

and

$$q : 3 > \left(\frac{7}{4}\right)^2$$

The premises of the argument are p and $p \to q$, and q is its conclusion. This argument is valid because it is constructed by using modes pones, a valid argument form. However, one of its premises $\sqrt{3} > 7/4$ is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, the conclusion of the argument is false, because $3 < 49/16$.

3.13.2 Law of Contraposition (Modus Tollens)

Here, the tautology is $(p \to q) \wedge {\sim}q \to {\sim}p$, which is based on the *law of contraposition* or *modus tollens* (modus tollens is a *Latin word* meaning *mode that denies*). The tautology is expressed in the following form:

$$\begin{array}{c} p \to q \\ {\sim}q \\ \hline \therefore {\sim}p \end{array}$$

The validity of modus tollens can be shown to follow from the modus pones together with the fact that a conditional statement is logically equivalent to its contrapositive.

(i) Here, $P(x)$ is false for every element in $P(x)$, still $P(x)$ is a propositional function on M.
(ii) $P(x)$ is the propositional function.
(iii) Let $3 + 2i$, $i = \sqrt{-1}$ is an imaginary number, be a complex number. It may be noted that $3 + 2i > 7$ does not have any meaning. In other words, inequalities are not defined for complex numbers.

3.16 QUANTIFIER

If in a propositional function the variables are prescribed, the resulting statement yields a proposition with a certain truth table. However, there is another way, called *quantification*, by which we can create a proposition from a propositional function. Quantification expresses a measure with which a predicate is true over a range of elements. In English literature, the words some, few, many, all and none are used in quantification. Here, we will discuss two types of quantification: universal which says that a predicate is true for every element under consideration, and existential quantification, which informs that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the *predicate calculus*.

3.16.1 Universal Quantifier

It has been observed that several mathematical statements affirm that a property is true for all values of a variable in a particular domain, called the *domain of discourse* (or the *universe of discourse*), and often just referred to as the *domain*. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts $P(x)$ is true for all values of x in this domain. The meaning of the universal quantification of $P(x)$ varies when we vary the domain. Without domain, the universal quantifications are undefined.

Consider the expression

$$(\forall x \in A)P(x) \quad \text{or} \quad \forall x P(x) \tag{3.1}$$

which reads 'for every x in A, $P(x)$ is a true statement', or simply, 'for all x, $P(x)$'. The symbol $\forall$ which reads 'for all' or 'for every' is called the *universal quantifier*. The statement (3.1) is equivalent to the statement

$$T_P = \{x : x \in A, P(x)\} = A \tag{3.2}$$

i.e., the truth set of $P(x)$ is the entire set A.

The expression $P(x)$ by itself is an open sentence or condition and therefore has no truth value. However, it is preceded by the quantifier, does have a truth value which follows from the equivalence of Eqs (3.1) and (3.2). Specifically,

$$Q_1\text{: If } \{x : x \in A, P(x)\} = A$$

then $\forall x\, P(x)$ is true; otherwise $\forall x\, P(x)$ is false.

Note An element for which $P(x)$ is false is called a *counter example* of $\forall x\, P(x)$.

For example

(i) The proposition $(\forall x \in N)\,(x + 4 > 3)$ is true since

$$\{x : x + 4 > 3\} = \{1, 2, 3, \ldots\} = N$$

(ii) The proposition $P(x)$: $x + 5 < 9$, then for all $x \geq 0$, is a false statement, since $P(5)$ is not true.

Example 3.48 Let $P(x)$ be the statement '$2x + 1 > 2x$'. What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution The quantification $\forall x\, P(x)$ is true because $P(x)$ is true for all real numbers x.

Example 3.49 Let $Q(x)$ be the statement '$x < 3$'. What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution The statement $Q(x)$ is not true for every real number x because, for instance, $Q(4)$ is false, i.e., $x = 4$ is a counter example for the statement $\forall x\, Q(x)$. Thus, $\forall x\, Q(x)$ is false.

Example 3.50 Consider the statement $\forall x\, N(x)$, where $N(x)$ is 'Computer is connected to the network'. What is the implication of the statement?

Solution The statement $\forall x\, N(x)$ means that for every computer x on campus, that computer x is connected to the network (see Table 3.32). This statement can be expressed in English as 'every computer on campus is connected to the network'.

3.16.2 Existential Quantifier

It is observed from the mathematical statement that there is an element with a certain property. Such statement can be expressed by prefixing $P(x)$ with the proposition 'there exists an element x'. The proposition 'there exists an x' is called an *existential quantifier*. The existential quantification of $P(x)$ is the proposition 'there exists a value of x' for which $P(x)$ is valid.

Consider the expression

$$(\exists x \in A)\, P(x) \quad \text{or} \quad \exists x,\, P(x) \tag{3.3}$$

which reads 'there exists an x in A such that $P(x)$ is a true statement' or, simply, 'for some x, $P(x)$'. The symbol $\exists$ which reads 'there exists' or 'for some' or 'for at least one' is called the *existential quantifier*. The statement (3.3) is equivalent to the statement

$$T_P = \{x : x \in A,\, P(x)\} \neq \phi \tag{3.4}$$

i.e., the truth set of $P(x)$ is not empty. Accordingly, $\exists x\, P(x)$, i.e., $P(x)$ is preceded by the quantifier $\exists$, does have a truth value. Specifically,

$$Q_2\text{: If } \{x : P(x)\} \neq \phi$$

then $\exists x\, P(x)$ is true; otherwise $\exists x\, P(x)$ is false.

For example

(i) The proposition $(\exists x \in N)(x + 4 < 7)$ is true, since $\{x : x + 4 < 7\} = \{1, 2\} = \phi$.

(ii) The proposition $(\exists x \in Z)(-1 < x < 1)$ is true, since $\{x : -1 < x < 1\} = \{-1, 0, 1\} = \phi$.

EXAMPLE 3.51 Let $P(x)$ denote the statement '$x > 3$'. Find the truth value of the quantific... $\exists x\, P(x)$ where the domain consists of all real numbers.

Solution Because '$x > 3$' is sometimes true; for instance, when $x = 4$, the exist... quantification of $P(x)$, which is $\exists x\, P(x)$ is also true.

EXAMPLE 3.52 Find the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement '$x^2 > 8$' and the universe of discourse consists of the positive integers not exceeding 3.

Solution Because of the given domain $\{1, 2, 3\}$, the proposition $\exists x\, P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3)$$

Thus, $P(3)$, which is the statement '$3^2 > 8$', is true; it follows that $\exists x\, P(x)$ is also true.

TABLE 3.32 Quantifier

Statement	True whenever	False whenever
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

3.16.3 Bound Variables

If a quantifier is applied on the variable x, then this occurrence of the variable is called *bound*. Also, an occurrence of a variable that is not bound by a quantifier or set is said to be *free*. The variables occurring in a propositional function must be bound or set equal to a particular value to convert it into a proposition. It can be made possible by the combination of universal quantifiers, existential quantifiers, and value assignments.

For example Let x and y be the variable in the statement $\exists x\,(x + y = 1)$. Actually, the variable x is bound by the existential quantification $\exists x$. However, the variable y is free, since it is not bound by a quantifier and no value is assigned to this variable. This shows that, in the statement $\exists x\,(x + y = 1)$, x is bound, but y is free.

7 INTRODUCTION TO PROOFS

In this section, we discuss the notion of a proof and describe methods for constructing proofs. A proof is a valid argument that establishes the truth of a mathematical statement.

The techniques of proof, we will discuss in this chapter, are important not only because they ...re used to prove mathematical theorems, but also for their many applications to computer sci... ...nce. These applications are used in verifying that computer programs are correct, establishing ...t operating systems are secure, making inferences in artificial intelligence, showing that sys... specifications are consistent, and so on. To understand the technique and its use in proofs ...ications are essential both in mathematics and in computer science.

$H_1, H_2, \ldots, H_n$ is *inconsistent* if their conjunction $H_1 \wedge H_2 \wedge \cdots \wedge H_n$ implies a contradiction, i.e., $H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow S \wedge \sim S$ (a contradiction), where S is any formula.

We use the conception of inconsistency in methods of proof, namely proof by contradiction and indirect proof with few examples.

3.17.5 Contraposition

Proofs of statement/theorems that are not direct, i.e., that do not start with the hypothesis and end with the conclusion, are called *indirect proofs*.

A versatile and a powerful type of indirect proof is known as *proof by contraposition*. This proof is based on the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive $\sim q \rightarrow \sim p$ is true. To prove by contraposition of $p \rightarrow q$, we first consider $\sim q$ as a hypothesis, and using axioms, definitions, and previously proven theorems, together the rules of inference, we show that $\sim p$ must follow.

Example 3.55 If k^2 is an even integer, then k is an even integer.

Solution Let p: k^2 is an even integer and q: k is an even integer. Assume first that $\sim q$ is true, then k is not an even integer. So, k must be odd, i.e., $k = 2m + 1$, for some integer m. Then

$$k^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$$

i.e., $k^2 = 2n + 1$, where $n = 2m^2 + 2m$. So, k^2 is odd and $\sim q \rightarrow \sim p$. Hence, by contraposition, k is an even integer.

3.17.6 Contradiction (reductio ad absurdum)

In this method of proof, we assume the negation of what we are trying to prove and get a logical contradiction. So, our assumption must have been false and what we were originally required to prove must be true. To prove $p \rightarrow q$ is true, we construct the procedure as follows:

(i) Assume $p \wedge (\sim q)$ is true.
(ii) Using the assumption, search some conclusion that is false.
(iii) The contradiction obtained in step (ii) leads us to the conclusion that $p \wedge (\sim q)$ is false which powers that $p \rightarrow q$ is true.

Example 3.56 Show that if $x^2 - 4 = 0$, then $x \neq 0$.

Solution Assume that $x = 0$, then $0^2 - 4 = -4 \neq 0$, which contradicts $x^2 - 4 = 0$. Hence, the assumption that $x = 0$ is false and we have proved that $x \neq 0$.

Example 3.57 Prove that $\sqrt{2}$ is not a rational number.

Solution Suppose that $\sqrt{2}$ is a rational number. Then, we can find integers a and b such that

$$\sqrt{2} = \frac{a}{b}$$

and we can assume that any common factors in a and b have been cancelled. We now square both sides of the equation to get

$$2 = \left(\frac{a}{b}\right)^2$$

Thus, $2b^2 = a^2$. Hence, a^2 is a multiple of 2, and is therefore even. So, a must be even, since the square of any odd number is also odd. Thus, 2 divides a and we can write $a = 2m$ for some integer m. Hence,

$$2b^2 = (2m)^2 \quad \text{or,} \quad 2b^2 = 4m^2 \quad \text{or,} \quad b^2 = 2m^2$$

Thus, b^2 is even. Hence, b is even. But now a and b have a common factor of 2, which is a contradiction to the statement that a and b have no common factors.

Hence, our initial assumption that $\sqrt{2}$ is a rational number is false. Thus, $\sqrt{2}$ is an *irrational* number.

3.17.7 Mathematical Induction

Statement: A sentence which can be judged to be true or false is called a *statement*. We generally denote a statement holding for $n \in N$ by $p(n)$

For example

(i) $p(n): 2^n$ is divisible by 2 for all $n \in N$.
It is clearly a true statement.

(ii) $p(n): (10n + 3)$ is prime.

Clearly, $p(3) = (10.3 + 3) = 33$, which is not prime. So, the given statement does not hold for all natural numbers.

In general, mathematical induction can be used to prove statements that assert that $p(n)$ is true for all positive integers n, where $p(n)$ is a propositional function.

To proof a mathematical statement (in the form of a formula) by mathematical induction, it requires two steps, a *basis step*, where we show that $p(1)$ is true, and an *inductive step*, where we show that for all positive integers k, if $p(k)$ is true, then $p(k + 1)$ is true.

Note In a proof by mathematical induction, it is only to be shown, if it is assumed that $p(k)$ is true, then $p(k + 1)$ is also true.

The *principle of mathematical induction* can be expressed as follows. Let $p(n)$ be a statement which is defined for the positive integers $n = 1, 2, 3, \ldots$. Then $p(n)$ is true for all positive integers n provided that

I. $p(1)$ is true.
II. $p(k + 1)$ is true whenever $p(k)$ is true.

Therefore, three steps are required to prove mathematical statement, using the principle of mathematical induction:

Step 1 (inductive basis step): Verify that $p(1)$ is true.
Step 2 (inductive hypothesis step): Assume that $p(k + 1)$ is true for an arbitrary value of k.
Step 3 (inductive step): Verify that $p(k + 1)$ is true on the basis of inductive hypothesis.

Note (replacement of basis step) The principle of mathematical induction chooses first $n = 1$ and then proves that $p(n)$ is true for $n \geq 1$. On the other hand, we can choose an integer different from 1, say, $n = p$ and prove that for $n = k + 1$ assuming that the statement is true for $n = k$ $(k \geq p)$.

The summation formula by mathematical induction can be proved as follows. Mathematical induction is well suited for proving the validity of summation formulae. However, the disadvantage of this method lies in the fact that we cannot use it to derive a summation formula. That means, you must have the formula with you before attempting to prove it by mathematical induction.

Example 3.58 Show that if n is a positive integer, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution Let $p(n)$ be the proposition or the statement that the sum of the first n positive integers

$$\frac{n(n+1)}{1}$$

We will then opt for two steps to prove that $p(n)$ is true for $n = 1, 2, 3, \ldots$. First of all, we will show that $p(1)$ is true and, secondly, that the conditional statement $p(k)$ implies that $p(k+1)$ is true for $k = 1, 2, 3, \ldots$.

Inductive basis step $p(1)$ is true because

$$1 = \frac{1(1+1)}{2}$$

Inductive hypothesis step Assume that $p(k)$ holds for an arbitrary positive integer k, i.e.,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Inductive step Under the assumption of inductive hypothesis step, it will be shown that $p(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true.

Adding $(k+1)$ to both sides of the equation in $p(k)$, we obtain

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

which shows that $p(k+1)$ *is true under the assumption that* $p(k)$ is true. Thus, by the use of *mathematical induction, it is shown that* $p(k)$ is true for all positive integer n.

Example 3.59 *Show by mathematical induction*

$$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution Let $p(n)$ be the given statement.

Inductive basis step For

$$n = 1, \frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$

i.e., $p(1)$ is true.

Inductive hypothesis step Assume that $p(k)$ is true, i.e.,

$$\frac{1}{1.2} + \frac{1}{1.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

is true.

Adding

$$\frac{1}{(k+1)(k+2)}$$

on both sides of $p(k)$, we obtain

$$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)(k+1)}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

Thus, $p(k+1)$ is true whenever $p(k)$ is true. Hence, by the principle of mathematical induction $p(n)$ is true for all positive integers n.

Example 3.60 Prove by induction that the expression for the number of diagonals in a polygon of n sides is

$$\frac{n(n-3)}{2}$$

Solution Given that there are n sides in a polygon. Among them consider two sides for the diagonals, possibly by ${}^nC_2$ ways.

Thus, the total number of diagonals is ${}^nC_2 - n$, i.e.,

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

Example 3.61 (extension of one of De Morgan's laws) Let $A_1, A_2, \ldots, A_n$ be any n sets. Show by mathematical induction that

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}$$

Solution Let $p(n)$ be the given statement.

Inductive basis step For $n = 1$, $p(1)$ is the statement $\overline{A_1} = \overline{A_1}$, which is true.

Inductive hypothesis step Assume that $p(k)$ is true, i.e.,

$$\overline{\left(\bigcup_{i=1}^{k} A_i\right)} = \bigcap_{i=1}^{k} \overline{A_i}$$

is true.

Inductive step Here, we will show that the statement $p(n)$ is true for $n = k + 1$. Then

$$\begin{aligned}
\overline{\left(\bigcup_{i=1}^{n+1} A_i\right)} &= \overline{A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}} \\
&= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \quad \text{(associative property of } \cup\text{)} \\
&= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k)} \cap \overline{A_{k+1}} \quad \text{(by De Morgans's law for two sets)} \\
&= \left(\bigcap_{i=1}^{k} \overline{A_i}\right) \cap \overline{A_{k+1}} \quad [\text{using } p(k)] \\
&= \left(\bigcap_{i=1}^{k+1} \overline{A_i}\right)
\end{aligned}$$

Thus, by the principle of mathematical induction $p(n)$ is true for all $n \geq 1$.

EXAMPLE 3.62 Show that the statement $2 + 4 + \cdots + 2n = (n + 2)(n - 1)$, for all $n \geq 1$ satisfies the inductive step but has no basis.

Solution Let $p(n)$ be the proposition that $2 + 4 + \cdots + 2n = (n + 2)(n - 1)$.

(i) $p(1)$ implies that $2 = (1 + 2)(1 - 1)$, which is not true.

(ii) If $p(n)$ were true, then $2 + 4 + \cdots + 2n = (n + 2)(n - 1)$ would be true, and by adding $2(n + 1)$ to both sides we would get

$$\begin{aligned}
2 + 4 + \cdots + 2n &= (n + 2)(n - 1) + 2(n + 1) \\
&= (n + 3)n = [(n + 1) + 2][(n + 1) - 1]
\end{aligned}$$

Hence, $p(n + 1)$ would also be true.

Thus, the inductive step is satisfied, but the basis for the induction fails and the result is false.

EXAMPLE 3.63 Let $N = n^2 + n + 41$. Show that there are some values of n for which N is a prime number, and others for which it is not. It follows that there is no inductive step which would show that $n^2 + n + 41$ is a prime number for all possible n.

Solution Let $p(n)$ imply that $n^2 + n + 41$ is a prime number.

(i) $p(1)$ implies that $1^2 + 1 + 41 = 43$ is a prime. This is true.

(ii) Now, we have to find a value of n for which $n^2 + n + 41$ is not a prime number! In fact, $p(1), p(2), \ldots, p(39)$ are all true, but $p(40)$ and $p(41)$ are false. For instance, if $n = 39$ then $N = 39^2 + 39 + 41 = 1601$, which is prime. However, if $n = 40$, then