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Principle of Inclusion and Exclusion

As we know the cardinality of the set P is the number of unique elements in set P. It is denoted as $|P|$ and read as cardinality of set P.

FIRST PRINCIPLE

If P and Q are disjoint sets, then

$$|P \cup Q| = |P| + |Q|.$$

Theorem I. *Let P and Q be any two non-disjoint sets. Then*

$$|P \cup Q| = |P| + |Q| - |P \cap Q|.$$

Proof. Draw Venn diagram for the above as shown in Fig. 1.

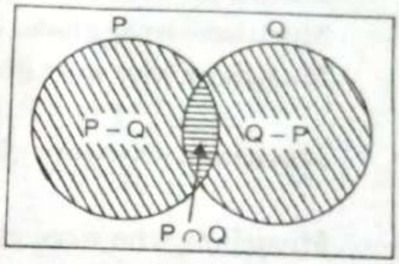

Fig. 1

From figure, we see that $P \cup Q$ can be seen to be the union of three disjoint sets $P - Q$, $Q - P$ and $P \cap Q$.

So $|P \cup Q| = |P - Q| + |Q - P| + |P \cap Q|$ ...(*i*)

Also, $|P| = |P - Q| + |P \cap Q|$ ...(*ii*)

and $|Q| = |Q - P| + |P \cap Q|$ ...(*iii*)

Combining (*ii*) and (*iii*)

$$|P| + |Q| = |P - Q| + |Q - P| + 2|P \cap Q|$$

$$|P| + |Q| = |P \cup Q| + |P \cap Q|$$

(As $|P \cup Q| = |P - Q| + |Q - P| + |P \cap Q|$)

$$|P \cup Q| = |P| + |Q| - |P \cap Q|$$

Hence proved.

Theorem II. *Let P, Q and R are three finite sets. Then*

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

Proof. Using theorem I, we have

$$|P \cup (Q \cup R)| = |P| + |Q \cup R| - |P \cap (Q \cup R)|$$

$$= |P| + |Q| + |R| - |Q \cap R| - |P \cap (Q \cup R)| \quad ...(i)$$

As $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

So $|P \cap (Q \cup R)| = |P \cap Q| + |P \cap R| - |(P \cap Q) \cap (P \cap R)|$

$$= |P \cap Q| + |P \cap R| - |P \cap Q \cap R| \quad ...(ii)$$

Putting (*ii*) in (*i*), we get

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap$$

Hence proved.

INCLUSION-EXCLUSION PRINCIPLE IN GENERAL

Let $P_1, P_2, \ldots\ldots, P_n$ are finite sets. Then $|P_1 \cup P_2 \cup \ldots\ldots P_n| = \sum_{1 \le i \le n} |P_i| - \sum_{1 \le i < j \le n} |P_i \cap$

$$+ \sum_{1 \le i < j < k \le n} |P_i \cap P_j \cap P_k| \ldots\ldots + (-1)^{n-1} |P_1 \cap P_2 \cap \ldots\ldots$$

SOLVED PROBLEMS

Problem 1. *In a survey of 200 musicians, it was found that 40 wore gloves on the hand and 39 wore gloves on the right hand. If 160 wore no gloves at all, how many wore a gl on only the right hand ? Only the left hand ? On both hands ?*

Sol. Total number of musicians wore gloves on left, right or both hands *i.e.*,

$$|L \cup R| = 200 - 160 = 40$$

Musicians wore gloves on left hand $|L| = 40$

Musicians wore gloves on right hand $|R| = 39$

Musicians who wore gloves on both hands

$$|L \cap R| = |L| + |R| - |L \cup R| = 40 + 39 - 40 = 39$$

Musicians who wore gloves only on right hand

$$= 40 - 39 = 1$$

Musicians who wore gloves only on left hand

$$= 39 - 39 = 0.$$

Problem 2. *Out of 1200 students at a college*

582 took Economics

627 took English

543 took Mathematics

217 took both Economics and English

307 took both Economics and Mathematics

250 took both Mathematics and English

222 took all three courses.

How many took none of the three ?

Sol. Suppose $|A| = 582$ $\quad |B| = 627$

$|C| = 543$ $\quad |A \cap B| = 217$

$|A \cap C| = 307$ $\quad |B \cap C| = 250$

$|A \cap B \cap C| = 222$

The total number of students who took any of three subjects

$$|A \cup B \cup C| = 582 + 627 + 543 - 217 - 307 - 250 + 222 = 1200$$

Students who took none of three

= (total students in the college) − (total students who took any of three subjects)

= 1200 − 1200 = 0.

Problem 3. *40 computer programmers interviewed for a job. 25 knew JAVA, 28 knew ORACLE, and 7 knew neither language. How many knew both languages ?*

Sol. Now

$$|J| = 25$$
$$|O| = 28$$
$$|J \cup O| = 40 - 7 = 33$$

Computer programmers who knew both languages are

$$|J \cap O| = 25 + 28 - 33 = 20.$$

Problem 4. *Among 100 students, 32 study Mathematics, 20 study Physics, 45 study Biology, 15 study Mathematics and Biology, 7 study Mathematics and Physics, 10 study Physics and Biology and 30 do not study any of three subjects.*

(a) Find the number of students studying all three subjects.

(b) Find the number of students studying exactly one of the three subjects.

Sol. $|M| = 32$ $\quad |P| = 20$ $\quad |B| = 45$

$|M \cap B| = 15$ $\quad |M \cap P| = 7$ $\quad |P \cap B| = 10$

$|M \cup P \cup B| = 100 - 30 = 70.$

(a) Number of students studying all three subjects

$$|M \cap P \cap B| = 70 - 32 - 20 - 45 + 15 + 7 + 10 = 5.$$

(b) 5 study all three subjects.

15 − 5 = 10 study Mathematics and Biology but not all the three.

7 − 5 = 2 study Mathematics and ~~Biology~~ Physics but not all the three.

10 − 5 = 5 study Physics and Biology but not all the three.

32 − 10 − 2 − 5 = 15 study only Mathematics.

20 − 2 − 5 − 5 = 8 study only Physics.

45 − 10 − 5 − 5 = 25 study only Biology.

Number of students studying exactly one of three subjects

= 15 + 8 + 25 = 48.

Problem 5. *A survey of 550 television watchers produced the following information :*

285 watch football games

195 watch hockey games

115 watch baseball games

45 watch football and baseball games

70 watch football and hockey games

50 watch hockey and baseball games

100 do not watch any of the three games.

(a) How many people in the survey watch all three games ?

(b) How many people watch exactly one of the three games ?

Sol.

$$|F| = 285 ; |H| = 195 ; |B| = 115$$
$$|F \cap B| = 45 ; |F \cap H| = 70 ; |H \cap B| = 50$$

Sol. Let A is the number of integers divisible by 5
B is the number of integers divisible by 7
C is the number of integers divisible by 9.

So $|A| = \left|\frac{1000}{5}\right| = 200$; $|B| = \left[\frac{1000}{7}\right] = 142$

$|C| = \left[\frac{1000}{9}\right] = 111$; $|A \cap B| = \left[\frac{1000}{5 \times 7}\right] = 28$

$|A \cap C| = \left[\frac{1000}{5 \times 9}\right] = 22$; $|B \cap C| = \left[\frac{1000}{7 \times 9}\right] = 15$

$|A \cap B \cap C| = \left[\frac{1000}{5 \times 7 \times 9}\right] = 3.$

(*a*) The number of integers divisible by 5, 7 and 9

$$|A \cup B \cup C| = 200 + 142 + 111 - 28 - 22 - 15 + 3$$
$$= 391.$$

The number of integers not divisible by 5, nor by 7, nor by 9

= Total number of integers – integers divisible by 5, 7 and 9

= 1000 – 391 = 609.

(*b*) The integers divisible by all the three integers = 3

28 – 3 = 25 integers divisible by 5 and 7 but not by all the three

22 – 3 = 19 integers divisible by 5 and 9 but not by all the three

∴ 200 – 25 – 19 – 3 = 153 integers divisible by 5 but not by 7, not by 9.