A clause is a disjunction ... set of m literals with ma

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Roll No...

Rawal Institute of Engineering & Technology, Faridabad

Pre – University Examination, April – 2024

Subject: Discrete Mathematics (PCC-CS_401) Branch – CSE, SEM – IV

Time: 3 Hours **Max. Marks: 75**

Instructions:

i. It is compulsory to answer all the questions (1.5 marks each) of Part – A in short.
ii. Answer any four from Part – B in detail.
iii. Different sub - parts of a question are to be attempted adjacent to each other.

PART – A (1.5x10=15)

1.

i. Define CNF.
ii. What are bijective functions?
iii. What is cut point and bridge in graphs?
iv. Define Normal Subgroup.
v. What is isomorphism of graph?
vi. What are quantifiers? Give example.
vii. Define Circular permutations.
viii. Show that the number of vertices having odd degree is always even in a graph.
ix. Prove using contraposition: if 3n+7 is an odd integer, then n is an even integer.
x. Define Euler Formula?

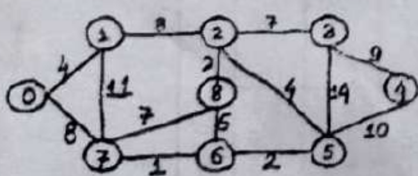
PART – B (15x4=60)

2.

a. Explain and prove Schroeder – Bernstein theorem.
b. What is a Perfect Graph? Explain with example.

3.

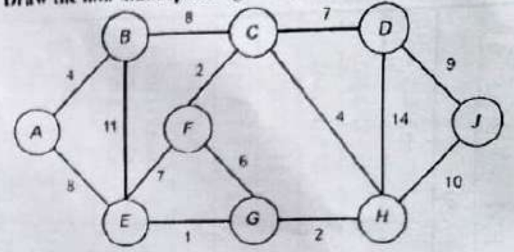
a. Find the shortest distance between 0 and 4:



b. If R is a relation on the set of positive integers I such that $(a, b) \in R$ if and only if $a - b = 1$. Determine whether the relation R is

i. An equivalence relation
ii. A partial order relation

a. Draw the minimum spanning tree for following graph.



b. Define Cantor's Diagonal Argument.

4.

a. What is Field in Algebraic systems? Explain with example.
b. Find whether the following implication is tautology, Contradiction or Contingency:

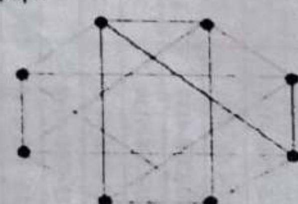
i. $(p \Rightarrow q) \vee r \Leftrightarrow [(p \vee r) \Rightarrow (q \vee r)]$
ii. $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$

5.

a. Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.
b. Consider an algebraic operation system $(G, *)$, where G is the set of all non – real numbers and $*$ is a binary operation defined by $a * b = \frac{ab}{4}$. Show that $(G, *)$ is an Abelian group.

6.

a. Find the validity of the following argument:
Robbery was the motive for the crime only if victim had money in his pockets. But robbery or vengeance was the motive for the crime. Therefore, vengeance must have been the motive for the crime.
b. What do you mean by the graph colouring and chromatic number of the graph? Determine the chromatic number of the following graph.



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Part-A

(i) Conjunctive Normal form

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a set of m literals with $m \in \mathbb{N}$.
A clause is a disjunction $\lambda_1 \vee \lambda_2 \vee \ldots \vee \lambda_m$ of m literals.
A formula ϕ is in CNF if it is a conjunction $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_K$ of K clauses $\phi_1, \phi_2, \ldots, \phi_K$ where $K \in \mathbb{N}$.

e.g. ① The disjunction of clauses shown is in CNF:
$(a \vee b \vee c) \wedge (\sim a \vee \sim b \vee \sim c) \wedge (a \vee \sim c \vee q)$

② $a \vee b \vee \sim c$ and p are in CNF. Each is a conjunction of one clause.

(ii) Bijective functions (Bijections):-
Let A and B be two non-empty sets. A function f from set A to set B is bijective if:

1. Every element in set A is mapped to a unique element in set B.

2. Every element in set B is mapped to by exactly one element in set A.

i.e. for every x in set A, there is exactly one y in set B such that $f(x) = y$, and for every y in set B, there is exactly one x in set A such that $f(x) = y$.

iii Cut point (articulation point) :-

Cut point is a vertex whose removal from the graph increases the number of connected components. i.e removing a cut point from the graph disconnects the graph, splitting it into two or more separate components.

Bridge (cut edge) :-

Bridge is an edge in a graph whose removal increases the number of connected components. Removing a bridge disconnets the graph, creating two separate components where there was previously one.

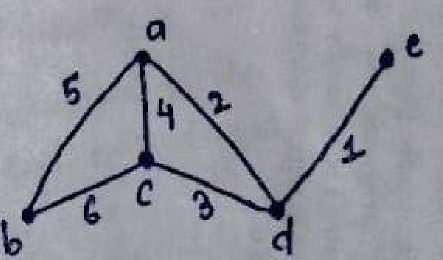
(iv) Normal subgroup :-

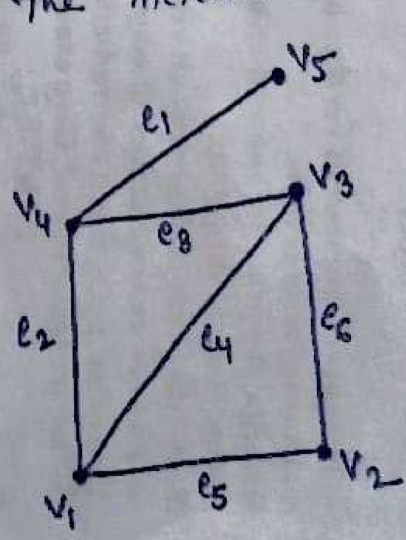
A subgroup H of a group G is said to be a normal subgroup of G if for every $x \in G$ and for every $h \in H$ $xhx^{-1} \in H$.

(v) Isomorphism of Graphs

Two graphs G and G' are said to be isomorphic if there is a one to one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

e.g.





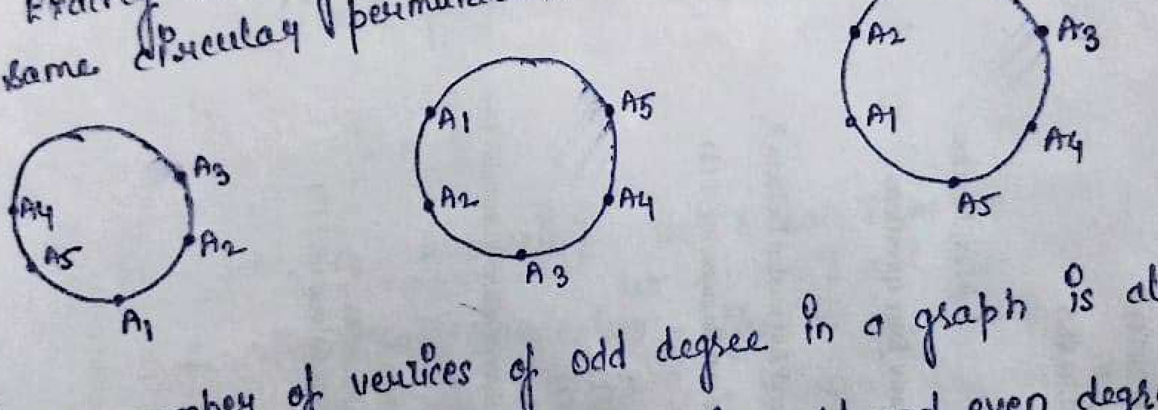
(vi) Quantifiers :-

The symbols $\forall$ and $\exists$ are called quantifiers, where $\forall$ is the universal quantifier and $\exists$ is the existential quantifier.

e.g $(\sim(\exists x\ P(x,y,z)))$, $(\forall y(\sim(\exists x\ P(x,y,z))))$ are formulas.

(vii) Circular permutations :-

Let S be a nonempty finite multiset. By a circular arrangement of the arrangement of elements of S, we mean an arrangement of the element of S on a circle. Two circular arrangements are the same if each element has the same 'clockwise adjacent' element i.e one can be obtained as a rotation of the other. Exactly two of the following pictures represent the same circular permutation.



(viii) The number of vertices of odd degree in a graph is always even

proof :- If we consider the vertices with odd and even degrees separately, the quantity in left side of $\sum_{i=1} d(v_i) = 2e$ can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k)$$

Since the left hand side in above is even and the first expression on the R.H.S is even, the second expression must-

$$\sum_{odd} d(v_k) = \text{an even number.}$$

Because in above each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number.

(ix) Original statement : If $3n+7$ is an odd integer, then n is an even integer.

Contrapositive : If n is not an even integer, then $3n+7$ is not an odd integer.

Assume that n is not an even integer, which mean it's an odd integer. Let's $n = 2k+1$, where k is an integer.

then $3n+7 = 3(2k+1)+7 = 6k+3+7 = 6k+10 = 2(3k+5)$

which is even.

Thus if n is not an even integer, then $3n+7$ is not an odd integer.

(x) Euler formula :- For any planar graph with v vertices, e edges and f faces, we have $v-e+f=2$.

Part - B

2 (i) [Cantor - Schroder - Bernstein] - Let X and Y be non empty sets and let $f: X \to Y$ and $g: Y \to X$ be one - one functions. Then there exists a bijection $h: X \to Y$.

proof :- If f is onto, then f itself is a bijection. So, assume that f is not onto. Then $f(X)$ is a proper subset of Y. Write $B = Y \setminus f(X)$, $\phi = f \circ g$ and $A = B \cup \phi(B) \cup \phi^2(B) \cup \cdots = B \cup \bigcup_{n=1}^{\infty} \phi^n(B)$.

Then $A \subseteq Y$ and $\phi(A) = \phi(B) \cup \bigcup_{n=2}^{\infty} \phi^n(B) = \bigcup_{n=1}^{\infty} \phi^n(B)$

Hence $A = B \cup \phi(A)$. Note that $f(X) = Y \setminus B$, $\phi(A) = f(g(A)) \subseteq Y$, and f is one-one. Hence

$f(X \setminus g(A)) = f(X) \setminus f(g(A)) = [Y \setminus B] \setminus \phi(A)$

$= Y \setminus [B \cup \phi(A)] = Y \setminus A$

Thus, the restriction of f to $X \setminus g(A)$ is a bijection onto $Y \setminus A$.

As g is one-one, its restriction to A is a bijection onto $g(A)$.

i.e $g^{-1}: g(A) \to A$ is a bijection. Therefore, the function $h: X \to Y$

defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in X \setminus g(A) \\ g^{-1}(x), & \text{if } x \in g(A) \end{cases}$$

is a bijection.

Q2

b. Perfect Graph :-

Given a graph $G = (V, E)$, $\chi(G)$ denotes the minimum number of colours required to properly colour all vertices of G and $\omega(G)$ denotes the size of the largest clique in G.

Then A graph $G = (V, E)$ is perfect if for all $S \subseteq V$,
$\omega(G[S]) = \chi(G[S])$.

e.g Any bipartite graph G is perfect.
This is trivial as any induced subgraph of a bipartite graph is bipartite and the largest clique in a bipartite graph is 2 while the number of colours needed is 2.

Q3 (a) let $G = (V, E)$. Here $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Taking $P_1 = \{0\}$, $T_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$l(1) = 4$, $l(2) = \infty$, $l(3) = \infty$, $l(4) = \infty$, $l(5) = \infty$, $l(6) = \infty$

$l(7) = 8$, $l(8) = \infty$

Here $1 \in T_1$, has minimum index $= 4$

Now Taking $P_2 = \{0, 1\}$, $T_2 = \{2, 3, 4, 5, 6, 7, 8\}$

$l(2) = \min(\infty, 12) = 12$

$l(3) = \min(\infty, \infty) = \infty$

$l(4) = \min(\infty, \infty) = \infty$

$l(5) = \min(\infty, \infty) = \infty$

$l(6) = \min(\infty, \infty) = \infty$

$l(7) = \min(8, 15) = 8$

$l(8) = \min(\infty, \infty) = \infty$

Here $7 \in T_2$, has minimum index $= 8$.

Now Taking $P_3 = \{0, 1, 7\}$ $T_3 = \{2, 3, 4, 5, 6, 8\}$

$l(2) = \min(12, \infty) = 12$

$l(3) = \min(\infty, \infty) = \infty$

$l(4) = \min(\infty, \infty) = \infty$

$l(5) = \min(\infty, \infty) = \infty$

$l(6) = \min(\infty, 9) = 9$

$l(8) = \min(\infty, 15) = 15$

Here $6 \in T_3$, has minimum index $= 9$.

Now taking $P_4 = \{0, 1, 7, 6\}$, $T_4 = \{2, 3, 4, 5, 8\}$

$l(2) = \min(12, \infty) = 12$

$l(3) = \min(\infty, \infty) = \infty$

$l(4) = \min(\infty, \infty) = \infty$

$l(5) = \min(\infty, 11) = 11$

$l(8) = \min(15, 15) = 15$

Here $5 \in \bar{P}_4$, has minimum index = 11

Now taking $P_5 = \{0,1,7,6,5\}$. $\bar{P}_5 = \{2,3,4,8\}$

$l(2) = \min(12, 15) = 12$

$l(3) = \min(\infty, 25) = 25$

$l(4) = \min(\infty, 21) = 21$

$l(8) = \min(15, \infty) = 15$

Here $2 \in \bar{P}_5$, has minimum index = 12

Now taking $P_6 = \{0,1,7,6,5,2\}$ $\bar{P}_6 = \{3,4,8\}$

$l(3) = \min(25, 19) = 19.$

$l(4) = \min(21, \infty) = 21$

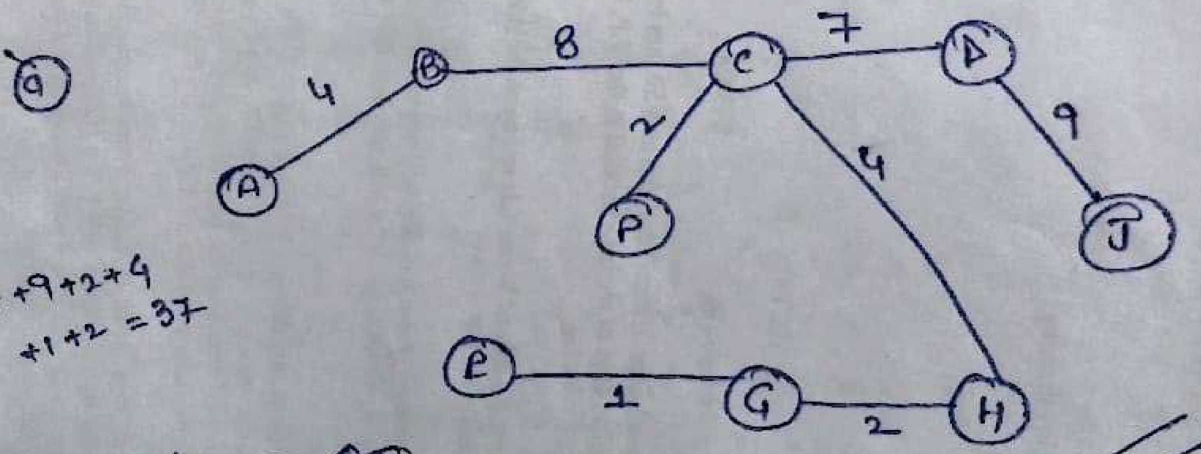
$l(8) = \min(15, 14) = 14$

~~Here $8 \in \bar{P}_6$ has minimum index = 14~~

~~$P_7 = \{0,1,7,6,5,2,8\}$ $\bar{P}_7 = \{3,4\}$~~

~~$l(3)$~~ Hence minimum distance from 0 to 4 is $\underline{\underline{21}}$

$\underline{\underline{4}}$



4+8+7+9+2+4
+1+2 = 37

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(b)

(i) Reflexivity :- for any positive integer a
$(a,a) \in R$. since $a-a=0 \leq 1$
$\therefore$ reflexivity holds.

(ii) Symmetry :- if $(a,b) \in R$ then (b,a) must also be in R

if $a-b \leq 1$ then $b-a \geq -1$

It is not always true that if $(a,b) \in R$ then $(b,a) \in R$

$\therefore$ symmetry doesn't hold.

(iii) Transitivity :- if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

if $a-b \leq 1$ and $b-c \leq 1$

$a-c = a-b+b-c \leq 1+1 \leq 2$

which still satisfies the condition.

$\therefore$ transitivity holds.

(iv) Antisymmetry :- If $(a,b) \in R$ and $(b,a) \in R$ then $a=b$.

If $a-b \leq 1$ and $b-a \leq 1 \Rightarrow a=b$

$\therefore$ Antisymmetry holds.

(I) R is not an equivalence relation

(II) R is a partial order relation.

Cantor's Diagonal Argument :-

Cantor found two important proof techniques for showing that two sets had the same cardinality.
The first of these arguments, called cantor's first diagonal argument, is used in proving that $|Z| = |Q|$

Q is countably infinite -

proof :- For each rational number r, pick out its expression p/q in lowest terms where $q > 0$.

For every rational number p/q written in lowest terms, computer the number $|p| + q$. Since p and q are integers and $q > 0$. $|p| + q$ is a positive integer. For $|p|+q = 1$ and 2, we have

$$\frac{0}{1}, \quad |p| + q = 1$$

$$-\frac{1}{1}, \frac{1}{1}, \quad |p| + q = 2$$

Then, for $|p| + q = 3$, we have $-\frac{2}{1}, -\frac{1}{2}, \frac{1}{2}, \frac{2}{1}$ (not in lowest terms), For $|p| + q = 4$, skipping $-2/2$ and $2/2$ We have $-\frac{3}{1}, \frac{3}{1}$. and so fourth.

The order in which the distinct rationals are listed is shown in fig. where the rationals p/q are represented as points on the plane with x coordinate p and y coordinate q.

order for listing elements of Q.

For any positive integer n, there are only finitely many different rational numbers p/q with $|p|+q=n$.

In fact, since q must be greater than zero, q must be one of $1,2,3,\ldots,n$ for a total of n choices.

There are two choices for p, $n-q$ and $-(n-q)$, giving a total of $2n$ choices for p/q. Among these $2n$ choices, some such as $0/2$ and $2/2$, will not be in lowest terms and so will be ignored. Then,

Let p/q be a rational number such that $|p|+q=n$. there are fewer than

$$2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \cdots + 2\cdot n = n\cdot(n+1)$$

rationals that could be listed in front of p/q on the list.

Hence, every rational number ultimately appears in lowest terms, each rational number is listed only once. Furthermore, since each rational number is listed only once. on the list above.

Set $G(n)$ to be n^{th} rational number on the list above.

Then, by the discussion above, $G: \mathbb{N} \to \mathbb{Q}$ is 1-1 and Onto.

Cantor's Second diagonal argument:

$\mathbb{R}$ is Uncountable.

4 a

Field – A ring R with at least two elements is called a field if it,

(i) is commutative

(ii) has Unity;

(iii) is such that each non-zero element possesses multiplicative inverse.

e.g. The set of numbers of the form $a+b\sqrt{2}$, with a and b as rational numbers is a field.

Sol :- $R=\{a+b\sqrt{2} : a,b \in Q\}$

Let $a_1+b_1\sqrt{2} \in R$ and $a_2+b_2\sqrt{2} \in R$. Then $a_1, b_1, a_2, b_2 \in Q$

We have $(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = (a_1+a_2)+(b_1+b_2)\sqrt{2}$ $\in R$ since $a_1+a_2, b_1+b_2 \in Q$

Also $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_1a_2+2b_1b_2)+(a_1b_2+a_2b_1)\sqrt{2} \in R$ $a_1a_2+2b_1b_2, a_1b_2+a_2b_1 \in Q$

Thus R is closed with respect to addition and multiplication. All the elements of R are real numbers and we know that addition and multiplication are both associative as well as commutative compositions in the set of real numbers.

Further $0+0\sqrt{2} \in R$, since $0 \in Q$

If $a+b\sqrt{2} \in R$ then $(0+0\sqrt{2})+(a+b\sqrt{2}) = (0+a)+(0+b)\sqrt{2} = a+b\sqrt{2}$

$\therefore$ $0+0\sqrt{2}$ is the additive identity.

Again if $a+b\sqrt{2} \in R$, then $(-a)+(-b)\sqrt{2} \in R$ and $[(-a)+(-b)\sqrt{2}]+[a+b\sqrt{2}] = 0+0\sqrt{2}$.

$\therefore$ each element of R possesses additive inverse.

Further in the set of real numbers multiplication is distributive with respect to addition.

(4) again $1+0\sqrt{2} \in R$ and

$(1+0\sqrt{2})(a+b\sqrt{2}) = a+b\sqrt{2} = (a+b\sqrt{2})(1+0\sqrt{2})$

$\therefore$ $1+0\sqrt{2}$ is the multiplicative identity.

Thus R is a commutative ring with unity.

The zero element of the ring is $0+0\sqrt{2}$ and the unit element is $1+0\sqrt{2}$.

Now R will be a field if each non-zero element of R possesses multiplicative inverse.

Let $a+b\sqrt{2}$ be any non-zero element of this ring

Then $\dfrac{1}{a+b\sqrt{2}} = \dfrac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \dfrac{a-b\sqrt{2}}{a^2-2b^2}$

$= \left(\dfrac{a}{a^2-2b^2}\right) + \left(-\dfrac{b}{a^2-2b^2}\right)\sqrt{2}$

Now if a and b are rational numbers, then we can have $a^2 = 2b^2$ only if $a=0$, $b=0$. Since here at least one of the rational numbers a and b is not 0, therefore we cannot have $a^2 = 2b^2$ i.e $a^2-2b^2 = 0$

$\therefore$ $\dfrac{a}{a^2-2b^2}$ and $\dfrac{-b}{a^2-2b^2}$ are both rational numbers and at least one of them is not zero.

$\therefore$ $\left(\dfrac{a}{a^2-2b^2}\right) + \left(-\dfrac{b}{a^2-2b^2}\right)\sqrt{2}$ is a non-zero element of R and is the multiplicative inverse of $a+b\sqrt{2}$.

Hence, the given system is a field.

b Do yourself

11.5

a :- $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Step 1 for $n = 1$

$1 + 2 = 2^{1+1} - 1 = 3.$

Hence statement is true for $n = 1$

Step 2 Let statement is true for $n = K$

$\therefore \; 1 + 2 + 2^2 + \cdots + 2^K = 2^{K+1} - 1$

Step 3 for $n = K+1$

Now $1 + 2 + 2^2 + \cdots + 2^K = 2^{K+1} - 1$ (from step 2)

$1 + 2 + 2^2 + \cdots 2^K + 2^{K+1} = 2^{K+1} - 1 + 2^{K+1}$

$= 2 \cdot 2^{K+1} - 1$

$= 2^{(K+1)+1} - 1$

$\therefore$ Statement is true for $n = K+1$

Hence, by Mathematical induction given statement is true.

6. $(G, *)$, where G is the set of all non-zero real number and $*$ is binary operation such that

$$a*b = \frac{ab}{4}$$

1. Closure :- $\forall \; a, b \in G$

$$a*b = \frac{ab}{4}$$ { Since the multiplication of any two non-zero real numbers is non-zero real number.

$\therefore \frac{ab}{4} \in G$

$\therefore$ G is closed with respect to $*$.

2. Associative $\forall \; a, b, c \in G$.

$$(a*b)*c = a*(b*c)$$

$$\left(\frac{ab}{4}\right)*c = a*\left(\frac{bc}{4}\right)$$

$$\frac{\frac{ab}{4}c}{4} = \frac{a\frac{bc}{4}}{4}$$

$$\frac{abc}{16} = \frac{abc}{16}$$ { $\because$ a,b,c are non-zero real No.

$\therefore$ G is associative with respect to $*$

3. Identity :- $\forall \; a \in G \; \exists \; 4 \in G$ such that

$$a*4 = a = 4*a.$$

$\therefore$ 4 is the identity element in G.

4. Inverse :- $\forall \; a \in G \; \exists \; \frac{16}{a} \in G$ such that

$$a * \frac{16}{a} = 4 = \frac{16}{a} * a.$$

$\therefore$ every element in G possesses inverse.

(5) Commutative $\forall\ a, b \in G.$

$$a * b = b * a$$

$$\frac{ab}{4} = \frac{ba}{4}$$ { since a & b are real & real No. are commutative }

hence $\frac{ab}{4} = \frac{ab}{4}$

$\Rightarrow G$ is commutative.

$\therefore (G, *)$ is an abelian group.

6 a let p = robbery was the motive for the crime

q = the victim had money in his pockets

r = Vengeance was the motive for the crime

Then the argument translates as follows:

$p \to q$

$p \vee r$

$\therefore r$

The truth table is :

p	q	r	$p \to q$	$p \vee r$	r
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	F

This is clearly not a valid argument as stated above, if the victim had money in their pocket and the motivation of the crime was robbery but not vengeance, this satisfies all hypothesis, but not the conclusion as suggested by the truth table.

b Do yourself -